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Preface

The Theory of Inequalities began its development from the time when C. F. GAUSS, A. L. CAUCHY and P. L. ČEBYŠEV, to mention only the most important, laid the theoretical foundation for approximative methods. Around the end of the 19th and the beginning of the 20th century, numerous inequalities were proved, some of which became classic, while most remained as isolated and unconnected results.

It is almost generally acknowledged that the classic work "Inequalities" by G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, which appeared in 1934, transformed the field of inequalities from a collection of isolated formulas into a systematic discipline. The modern Theory of Inequalities, as well as the continuing and growing interest in this field, undoubtedly stem from this work. The second English edition of this book, published in 1952, was unchanged except for three appendices, totalling 10 pages, added at the end of the book.

Today inequalities play a significant role in all fields of mathematics, and they present a very active and attractive field of research.

J. DIEUDONNÉ, in his book "Calcul Infinitésimal" (Paris 1968), attributed special significance to inequalities, adopting the method of exposition characterized by "majorer, minorer, approcher".

Since 1934 a multitude of papers devoted to inequalities have been published: in some of them new inequalities were discovered, in others classical inequalities were sharpened or extended, various inequalities were linked by finding their common source, while some other papers gave a large number of miscellaneous applications.

The book "Inequalities" by E. F. BECKENBACH and R. BELLMAN, which appeared in 1961 and the second revised printing in 1965, contains an account of some results on inequalities obtained in the period 1934—1960.

The present book — "Analytic Inequalities" — is devoted for the most part to topics which are not included in the two mentioned above. However, even in the exposition of classical inequalities new facts have been added.

We have done our best to be as accurate as possible and have given all the relevant references we could. A systematic bibliographical search was undertaken for a large number of inequalities, and we believe the results included are up to date.

In writing this book we have consulted a very extensive literature. It is enough to mention that "Analytic Inequalities" cites over 750 names, some several times. As a rule, we have studied the original papers and only exceptionally have we leaned on the reviews published in Jahrbuch über die Fortschritte der Mathematik (1868—1944), Zentralblatt für Mathematik (since 1931), Mathematical Reviews (since 1940) and Referativnyĭ Žurnal Matematika (since 1953). Nevertheless, it was impossible to scan every relevant source and, for various reasons, some omissions were inevitable; we apologize in advance to anyone whose work may not have been given proper credit through oversight. Besides, our selection from the enormous material considered expresses our preference for simple and attractive results.

The greater part of the results included have been checked, although this could not, of course, be done for all the results which appear in the book. We hope, however, that there are not many errors, but the very nature of this book is such that it seems impossible to expect it to be entirely free of them. It is perhaps unnecessary to point out the advisability of checking an inequality before use. It is also worthwhile to turn to the original papers whenever possible, since the reader will frequently find the problem which first motivated the search for the inequality in question.

Though we have emphasized only in a relatively few places that there are unsolved problems, it can be seen from the text itself that there are many results which can be improved or developed in various directions.

This book is, in fact, a considerably extended and improved version of the author's book "Nejednakosti" which appeared in Serbian in 1965. Although following the idea and the outline of that work, "Analytic Inequalities" is on a higher level, and contains very little of the same material. It also contains many inequalities which are now published for the first time, owing to the fact that many mathematicians generously offered us their unpublished results.

The material of this book is divided into three parts. In the first part — "Introduction" — an approach to inequalities is given, while the main attention is devoted to the Section on Convex Functions.

The second and probably main part — "General Inequalities" — consists of 27 sections, each of which is dedicated to a class of inequalities of importance in Analysis. Special attention was paid to some sections, for instance to Sections 2.11, 2.14, 2.16, 2.23, 2.25, and we believe that they will be of benefit for further research.

Finally, the third part — "Particular Inequalities" — is aimed at providing a collection of various inequalities, more or less closely interconnected, some of which are of considerable theoretical interest. They are classified in a certain manner, although we must admit that this has not been done perfectly. Part 3 is, in fact, a collection of over 450 special inequalities, and with a few exceptions we were able to add bibliographical references for each one. Owing to lack of space, only a few inequalities are supplied with a complete proof.

As may be inferred from the title — "Analytic Inequalities" — various topics such as geometric inequalities, isoperimetric inequalities, as well as inequalities arising in Probability Theory have not been included. We have also omitted inequalities for univalent and multivalent functions, inequalities arising in Number Theory, inequalities which belong to the Theory of Forms, inequalities such as BESSEL's inequality, which belong to the Theory of Orthogonal Series, as well as inequalities arising in the Theory of Special Functions.

This book could be used as a postgraduate reference book, but undergraduate students may also successfully consult individual sections of it. Naturally "Analytic Inequalities" will be of use to those researching in the Theory of Inequalities, but we believe it will also prove useful to mathematicians, engineers, statisticians, physicists and all who come across inequalities in their work.

If it is true that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove", we may expect that "Analytic Inequalities" will be of some help to them.

A large number of inequalities also hold under weaker conditions than those given here. This is especially true of inequalities involving integrals or positive integers.

It is a shortcoming of the book that the conditions under which strict inequalities hold are not specified everywhere. The form of exposition is not uniform throughout Part 2, in which the majority of results are stated as theorems with proofs, while the others are given more descriptively without emphasizing the theorem.

In the final phase of composing this book, Assistant Professor P.M. VASIĆ gave a considerable help in the classification of various topics, in writing some individual sections or subsections, as well as in the critical review of almost the whole text, and for these reasons his name appears on the title page.

Professor P.S. BULLEN of Vancouver University (Canada) wrote the Section 2.15 on Symmetric Means and Functions which is included here with some minor changes and additions.

Professor P.R. BEESACK of Ottawa University (Canada) was good enough to read the whole manuscript as well as the proofs. His remarks, suggestions and comments were very helpful.

Individual sections or subsections of "Analytic Inequalities" were kindly read by Professors J. ACZÉL, P.S. BULLEN and D. Ž. DJOKOVIĆ from Canada, Professors ROY O. DAVIES, H. KOBER, L. MIRSKY and R.A.

RANKIN from England, Professor M. JANET from France, Professor E. MAKAI from Hungary, Professors J. MUSIELAK and Z. OPIAL from Poland, Professors S. MARCUS and T. POPOVICIU from Romania, Professor P. H. DIANANDA from Singapore, Professor A. V. BOYD from South Africa, Professors R. P. BOAS, K. FAN, D. C. B. MARSH, J. RYFF and O. TAUSSKY-TODD from the U.S.A., and Professors S. KUREPA, M. MARJANOVIĆ and S. B. PREŠIĆ from Yugoslavia. The author greatly profited by their criticism, comments and suggestions.

Without their assistance many misprints and even errors would have probably remained unnoticed. In addition, some sections or subsections have been largely rewritten as a result of BEESACK's and KUREPA's suggestions.

The young Yugoslav mathematician J. D. KEČKIĆ not only helped with the translation of the manuscript into English, but also gave valuable comments on the text itself. He also compiled the subject index.

Dr. R. R. JANIĆ assisted in collecting documentational material. Dr. D. DJ. TOŠIĆ, D. V. SLAVIĆ and M. D. MITRINOVIĆ helped in the technical preparation of the manuscript for print.

The author feels indebted to all those mentioned above for the help which they have, in one way or another, given him.

The author is also indebted to a number of mathematicians and institutions for their extremely valuable assistance in furnishing the necessary literature and regrets his inability to quote all of them.

The author will be obliged to readers for further bibliographical data and also for any comments on the content and form of this book. The author believes that it can be improved in various directions. Such comments would be especially valuable as the author, with several associates, is preparing a series of books treating individual classes of inequalities as, for example, integral inequalities, inequalities involving polynomials, trigonometric inequalities, inequalities involving special functions, etc.

The author wishes to thank Springer-Verlag for publishing his book in their distinguished series "Grundlehren der mathematischen Wissenschaften" and for their readiness to meet all requests.

Finally, we list the main books and sources related to inequalities in various directions:

1. HARDY, G. H., J. E. LITTLEWOOD and G. PÓLYA: Inequalities. Cambridge 1934, 314 pp.

Russian translation by V. I. LEVIN, with Supplements by V. I. LEVIN and S. B. STEČKIN, Moscow 1948, 456 pp. (Supplements: pp. 361—441).

A selection of the above Supplements by R. P. BOAS appeared under the title: LEVIN, V. I., and S. B. STEČKIN: Inequalities. Amer. Math. Soc. Transl. (2) 14, 1—29 (1960).

Second English edition 1952, 324 pp.

Chinese translation of the second English edition. Peking 1965, X + 352 pp.

2. PETROVIĆ, M.: Računanje sa brojnim razmacima. Beograd 1932, 193 pp.; 2nd ed. 1969, 169 pp.
3. PÓLYA, G., and G. SZEGÖ: Isoperimetric Inequalities in Mathematical Physics. Princeton 1951, XVI + 279 pp.
4. NATANSON, I. P.: Konstruktive Funktionentheorie. Berlin 1955, XIV + 515 pp.
5. AHIEZER, N. I.: Theory of Approximation. New York 1956, X + 307 pp.
6. TIMAN, A. F.: Theory of Approximation of Functions of a Real Variable (Russian). Moscow 1960, 624 pp.
7. BECKENBACH, E. F., and R. BELLMAN: Inequalities. Berlin-Heidelberg-NewYork 1961; 2nd ed. 1965, 198 pp.
8. Studies in Mathematical Analysis and Related Topics. Stanford 1962, 447 pp.
9. MARÇUS, M., and H. MINC: A Survey of Matrix Theory and Matrix Inequalities. Boston 1964, 180 pp.
10. MITRINOVIĆ, D. S.: Elementary Inequalities. Groningen 1964, 159 pp.
11. WALTER, W.: Differential- und Integral-Ungleichungen. Berlin-Göttingen-Heidelberg-New York 1964, XIII + 269 pp.
12. MITRINOVIĆ, D. S.: Nejednakosti. Beograd 1965, 240 pp.
13. HARDY, G. H.: Collected Papers, vol. 2. Oxford 1967, pp. 379—682.
14. Inequalities (Proceedings of a Symposium held at Wright-Patterson Air Force Base, Ohio 1965), edited by O. SHISHA. New York-London 1967, 360 pp.
15. SZARSKI, J.: Differential Inequalities. Warszawa 1967, 256 pp.
16. MITRINOVIĆ, D. S., and P. M. VASIĆ: Sredine. Beograd 1969, 122 pp.
17. BOTTEMA, O., R. Ž. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ and P. M. VASIĆ: Geometric Inequalities. Groningen 1969, 151 pp.

Belgrade, May 1970 D. S. MITRINOVIĆ

Organization of the Book

Besides the Preface, Notations and Definitions, and the Indexes, the book contains three parts, each of which is divided into a number of sections, and some of these into subsections. The numeration of theorems, definitions, remarks and formulas is continuous throughout a subsection, or a section which does not contain subsections. If the theorem referred to belongs to the same subsection, only its number is given, while if it belongs to another, the numbers of the part, section, subsection and of the theorem are given. Similar notations are used if a section is not divided into subsections.

There are many cross-references in the book. So, for example, 2.1.4 means Part 2, Section 1, Subsection 4.

As a rule, bibliographical references are quoted after each subsection if it exists, or after each section, if it does not. Sections 1.1, 1.4, 2.15 and 2.25 present exceptions to this rule.

The abbreviations of the cited journals are given according to Mathematical Reviews.

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On Notations and Definitions

The notations and concepts used throughout the book are more or less specified. The reader is assumed to be familiar with the elements of Mathematical Analysis and with the basic concepts of General Algebra and Topology, and since the standard notations were used, it was believed unnecessary to define all of them. We shall, therefore, list only a few of them.

$[x]$ denotes the integral part of the real number x.

If $a > 0$, and if p/q is any rational number, with p and q both integers and $q > 0$, then $a^{p/q}$ means the unique positive q-th root of a^p.

If r is a real number, then $(f(x))^r$ is often denoted by $f(x)^r$, and $(f^{(k)}(x))^r$ by $f^{(k)}(x)^r$.

If A and B are two arbitrary sets, then the set $A \times B$ is defined by

$$A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}.$$

R^n denotes the n-dimensional vector space of points x with coordinates $x_1, \ldots, x_n$. According to whether the coordinates are real or complex, R^n is called the real or the complex n-dimensional space.

A vector or a sequence is called positive (negative) if all its coordinates are positive (negative).

The scalar, or the inner, product of two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ is the number $(a, b) = a \cdot b = a_1\bar{b}_1 + \cdots + a_n\bar{b}_n$, where $\bar{b}_1, \ldots, \bar{b}_n$ denotes the complex conjugates of $b_1, \ldots, b_n$.

For two sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ we define the r sum and product as follows:

$$a + b = (a_1 + b_1, \ldots, a_n + b_n), \quad a \cdot b = (a_1 b_1, \ldots, a_n b_n).$$

Their difference and quotient are defined analogously, provided in the latter case that $b_i \neq 0$ for $i = 1, \ldots, n$.

$C[a, b]$ denotes the set of all real or complex functions continuous on the interval $[a, b]$.

$L^p[a, b]$ denotes the set of all real or complex functions f such that f^p is integrable on $[a, b]$.

$||f||$ denotes the norm of f with respect to a certain space.

If $A = (a_{ij})$ is an $n \times n$ matrix, then $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

1. Introduction

1.1 Real Number System

1.1.1 Axioms of the Set of Real Numbers

A systematic and a detailed construction of the real number system can be found, for example, in the book [1] of E. LANDAU, or in the book [2] of L. W. COHEN and G. EHRLICH.

We shall, in a manner similar to that of J. DIEUDONNÉ [3], give definitions and the system of axioms of the set of real numbers together with a number of theorems which follow directly from these axioms. Their proofs are more or less simple and will be omitted.

The set of real numbers is a nonempty set R together with two mappings

$$(x, y) \mapsto x + y \quad \text{and} \quad (x, y) \mapsto xy$$

from $R \times R$ into R, called addition and multiplication respectively, and an order relation $x \leq y$ (also written $y \geq x$) between elements of R so that:

1° R is a field,

2° R is an ordered field,

3° R is an Archimedean ordered field,

4° R is complete, i.e., R satisfies the axiom of nested intervals.

Since this monograph deals with inequalities, we shall consider in some detail only the order properties.

1.1.2 Order Properties of Real Numbers

In all relations below x, y, z are arbitrary elements of R.

By "R is an ordered field" we mean that the following axioms are satisfied:

2.1° $x \leq y$ and $y \leq z$ imply $x \leq z$;

2.2° $x \leq y$ and $y \leq x$ is equivalent to $x = y$;

2.3° for any x and y, either $x \leq y$ or $y \leq x$;

2.4° $x \leq y$ implies $x + z \leq y + z$;

2.5° $0 \leq x$ and $0 \leq y$ imply $0 \leq xy$.

The relation "$x \leq y$ and $x \neq y$" is written $x < y$, or $y > x$. The relation $x \leq y$ is equivalent to "$x < y$ or $x = y$".

Let $a < b$. The set $\{x \mid a < x < b\}$ is called the open interval with end-points a and b, and written (a, b). The set $\{x \mid a \leq x \leq b\}$ is called the closed interval or the segment with end-points a and b, and written $[a, b]$. For $a = b$ the notation $[a, a]$ denotes the one-point set $\{a\}$. By $[a, b)$ and $(a, b]$ we denote the sets $\{x \mid a \leq x < b\}$ and $\{x \mid a < x \leq b\}$ respectively, and these are called semi-open intervals.

Using the axioms 1°—4° in 1.1.1, and in particular, using the axioms given under 2.1°—2.5° we can prove the following important theorems.

Theorem 1. *For any $x, y \in R$ one and only one of the three relations $x < y$, $x = y$, $x > y$ holds.*

Theorem 2. *If "$x \leq y$ and $y < z$", or "$x < y$ and $y \leq z$", then $x < z$.*

Theorem 3. *Any finite subset A of R has the greatest element b and the smallest element a, thus $a \leq x \leq b$ for every $x \in A$.*

Theorem 4. *If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two finite sequences both of n real numbers, such that $x_k \leq y_k$ for $k = 1, \ldots, n$, then*

$$x_1 + \cdots + x_n \leq y_1 + \cdots + y_n.$$

If, in addition, $x_k < y_k$ for at least one index k, then

$$x_1 + \cdots + x_n < y_1 + \cdots + y_n.$$

A real number x is called positive if $x > 0$; negative if $x < 0$. A real number x is called nonnegative if $x \geq 0$, and nonpositive if $x \leq 0$. If $x > 0$ and $y > 0$, or if $x < 0$ and $y < 0$, we say that x and y are of the same sign. If $x > 0$ and $y < 0$, or if $x < 0$ and $y > 0$, we say that x and y have opposite signs.

Theorem 5. *If $x_1, \ldots, x_n$ is a sequence of n real numbers and if $y_1, \ldots, y_n$ is a sequence of n nonnegative real numbers such that $y_k \leq x_k$ for $k = 1, \ldots, n$, then*

$$y_1 \cdots y_n \leq x_1 \cdots x_n.$$

Theorem 6. *If $x + z \leq y + z$ $(x + z < y + z)$ for at least one $z \in R$, then $x \leq y$ $(x < y)$.*

Theorem 7. *The relations* $x \leq y$, $0 \leq y - x$, $x - y \leq 0$, $-y \leq -x$ *are equivalent. Same results hold if* $\leq$ *is replaced by* $<$.

For an interval (a, b), with $a < b$, the positive number $b - a$ is called the length of the interval.

Theorem 8. *Let* $J_1, \ldots, J_n$ *be* n *disjoint intervals, and let* I *be an interval containing* $\bigcup_{k=1}^{n} J_k$. *Then, if* l_k *is the length of* J_k, *for* $k = 1, \ldots, n$, *and if* l *is the length of* I,

$$l_1 + \cdots + l_n \leq l.$$

For any real number x, we define

$$|x| = x \text{ for } x \geq 0, \text{ and } |x| = -x \text{ for } x \leq 0.$$

Hence $|x| = \max(x, -x)$ and $|-x| = |x|$.
$|x|$ is called the absolute value of x.
$|x| = 0$ is equivalent to $x = 0$.
For $x \neq 0$ we write

$$x^+ = \frac{1}{2}(|x| + x) \quad \text{(positive part of } x\text{)},$$

$$x^- = \frac{1}{2}(|x| - x) \quad \text{(negative part of } x\text{)}.$$

We also write $0^+ = 0^- = 0$.
Using the above notations we have

$$x^+ = x \text{ if } x \geq 0, \quad x^+ = 0 \text{ if } x \leq 0;$$

$$x^- = 0 \text{ if } x \geq 0, \quad x^- = -x \text{ if } x \leq 0;$$

$$x = x^+ - x^-, \quad |x| = x^+ + x^-.$$

Theorem 9. *If* $a > 0$, *then* $|x| \leq a$ *is equivalent to* $-a \leq x \leq a$, *and* $|x| < a$ *to* $-a < x < a$.

Theorem 10. *For any pair* x, y *of real numbers,*

$$(1) \qquad |x + y| \leq |x| + |y|,$$

$$(2) \qquad \big||x| - |y|\big| \leq |x + y|,$$

and, by induction,

$$(3) \qquad |x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|.$$

Equality in (1) holds if and only if $x = 0$, or $y = 0$, or if x and y have the same sign.

Equality in (2) holds if and only if $x = 0$, or $y = 0$, or if x and y have opposite signs.

Equality in (3) holds if and only if all the numbers $x_1, \ldots, x_n$ not equal to zero have the same sign.

Theorem 11. *For any real numbers x, y*

$$(|x| - |y|)^2 \leq |x^2 - y^2|,$$

with equality if and only if $x = 0$, or $y = 0$, or if the absolute values of x and y are equal;

$$\left|\sqrt{|x|} - \sqrt{|y|}\,\right| \leq \sqrt{|x - y|},$$

with equality if and only if $x = 0$, or $y = 0$, or $x = y$.

Theorem 12. *If $z \geq 0$, then $x \leq y$ implies $xz \leq yz$.*

Theorem 13. *The relations $x \leq 0$ and $y \geq 0$ imply $xy \leq 0$. The relations $x \leq 0$ and $y \leq 0$ imply $xy \geq 0$. Same results hold with $\leq$ replaced by $<$. In particular, $x^2 \geq 0$ for any real number and $x^2 > 0$ unless $x = 0$.*

Theorem 14. *If $x > 0$, then $1/x > 0$. If $z > 0$, then $x \leq y$ is equivalent to $xz \leq yz$. The relation $0 < x < y$ is equivalent to $0 < 1/y < 1/x$.*

A real number b is said to be a majorant (resp. minorant) of a subset X of the set R if $x \leq b$ (resp. $b \leq x$) for every $x \in X$. A set $X \subset R$ is said to be majorized, or bounded from above (resp. minorized, or bounded from below) if the set of majorants (resp. minorants) of X is not empty. If X is majorized, then $-X = \{-x \mid x \in X\}$ is minorized, and for every majorant b of X, $-b$ is a minorant of $-X$, and vice versa. A set which is both majorized and minorized is said to be bounded.

Theorem 15. *For any real number $a > 0$ and $a \neq 1$, there exists the function $x \mapsto a^x$ from R onto the set of all positive numbers such that:*

1° $a^{x+y} = a^x \cdot a^y$ *for any* $x, y \in R$,

2° $a^0 = 1$,

3° $x < y \Leftrightarrow a^x < a^y$ *if* $a > 1$, *and* $x < y \Leftrightarrow a^x > a^y$ *if* $0 < a < 1$.

The inverse of the function $x \mapsto a^x$ is denoted by $x \mapsto \log_a x$ and called the logarithmic function.

Remark. There is the unique number $e > 1$ such that $(e^x)' = e^x$ for any $x \in R$. This number e is transcendental; $e = 2.7182818284\ldots$ The function $x \mapsto e^x$ is called the exponential function and sometimes denoted by $\exp x$. In fact $e = \lim_{x \to 0} (1 + x)^{1/x}$.

Theorem 16. *If $a \geq b > 0$ and $x \geq 0$, then*

$$a^x \geq b^x.$$

Equality holds if and only if $a = b$ or $x = 0$.

Theorem 17. *Let $a < b$ and let f and g be two real and piecewise continuous functions on $[a, b]$ such that $f(x) \leq g(x)$ for all the points of continuity of f and g (except, perhaps, in a finite number of points). Then*

$$\int_a^b f(t)\, dt \leq \int_a^b g(t)\, dt.$$

Equality holds if and only if $f(x) = g(x)$ in all the points of continuity of f and g.

In particular, if f is a piecewise continuous function on $[a, b]$, $f(x) \geq 0$, except at the points of discontinuity of f and $\int_a^b f(t)\, dt = 0$, then $f(x) = 0$ on $[a, b]$ except at the points of discontinuity.

Concerning Theorem 17, see, for example, the book [4] of J. DIEUDONNÉ.

References

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3. DIEUDONNÉ, J.: Foundations of Modern Analysis. New York-London 1960.
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1.2 Complex Number System

The field of complex numbers is defined as the set $R \times R$ of all ordered pairs (x, y) of real numbers with

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{as addition},$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \quad \text{as multiplication}.$$

Any complex number $z = (x, y)$ is written in the form $z = x + iy$, where $i = (0, 1)$, and any real number x is identified with the complex number $(x, 0)$.

For $z = x + iy$, the complex conjugate of z is defined by $\bar{z} = x - iy$. We have $z\bar{z} = x^2 + y^2$, and the nonnegative number $|z| = \sqrt{x^2 + y^2}$ is called the modulus of z.

Using the notations

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}),$$

we get

$$(1) \qquad \begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2 \\ &= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1\bar{z}_2), \end{aligned}$$

where z_1 and z_2 are arbitrary complex numbers.

Since

$$\operatorname{Re} z \leq |z| = [(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2]^{1/2}, \tag{2}$$

we have

$$\operatorname{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2| = |z_1|\,|z_2|. \tag{3}$$

From (1) and (3) follows

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| = (|z_1| + |z_2|)^2$$

i.e.,

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{4}$$

Equality holds in (4) if and only if there is an equality in (3). Since in (2) there is equality if and only if z is real and nonnegative, we infer that $z_1\bar{z}_2 \geq 0$ is necessary and sufficient for equality to hold in (3), and also in (4).

If z_1 and z_2 are nonzero, the condition $z_1\bar{z}_2 > 0$ is equivalent to $|z_2|^2\frac{z_1}{z_2} > 0$, i.e., $\frac{z_1}{z_2} > 0$. This has the following geometric interpretation in the complex plane: the points z_1 and z_2 lie on the same ray issuing from the origin.

Inequality (4) is usually called the triangle inequality. The following proposition which gives the conditions of equality in the triangle inequality is also important:

In the case $z_1 \neq 0$ and $z_2 \neq 0$ equality holds in (4) if and only if $z_2 = tz_1$ with $t > 0$.

Let us prove that

$$|z_1 - z_2| \geq \big|\,|z_1| - |z_2|\,\big|. \tag{5}$$

From $z_1 = (z_1 - z_2) + z_2$, using the triangle inequality (4), we find

$$|z_1| - |z_2| \leq |z_1 - z_2|. \tag{6}$$

Similarly, we obtain

$$|z_2| - |z_1| \leq |z_1 - z_2|. \tag{7}$$

Inequality (5) follows from (6) and (7).

Replacing z_2 with $-z_2$ in (5), we have

$$|z_1 + z_2| \geq \big|\,|z_1| - |z_2|\,\big|.$$

Combining this and the triangle inequality, we find

$$\big|\,|z_1| - |z_2|\,\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

We shall now examine the conditions of equality in (5), i.e., when

$$\big|\,|z_1| - |z_2|\,\big| = |z_1 + z_2| \tag{8}$$

holds.

If (8) is valid, then

$$(|z_1| - |z_2|)^2 = |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\bar{z}_2),$$

and so

$$\mathrm{Re}\,(z_1\bar{z}_2) = -|z_1|\,|z_2| = -|z_1\bar{z}_2|.$$

Hence it follows that $\mathrm{Re}\,(z_1\bar{z}_2) \leq 0$, and that

$$\big(\mathrm{Re}\,(z_1\bar{z}_2)\big)^2 = \big(\mathrm{Re}\,(z_1\bar{z}_2)\big)^2 + \big(\mathrm{Im}\,(z_1\bar{z}_2)\big)^2,$$

i.e., $\mathrm{Im}\,(z_1\bar{z}_2) = 0$. Thus $z_1\bar{z}_2$ is a real nonpositive number, i.e., $z_1\bar{z}_2 \leq 0$. Conversely, if $z_1\bar{z}_2 \leq 0$ we have

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\bar{z}_2) \\ &= |z_1|^2 + |z_2|^2 + 2\,z_1\bar{z}_2 \\ &= |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2| \\ &= (|z_1| - |z_2|)^2 \end{aligned}$$

and hence we get (8). Consequently, (8) is equivalent to $z_1\bar{z}_2 \leq 0$.

If z_1 and z_2 are nonzero, the condition $z_1\bar{z}_2 < 0$ is equivalent to $|z_2|^2 \frac{z_1}{z_2} < 0$, i.e., $\frac{z_1}{z_2} < 0$. This has the following geometric interpretation in the complex plane: the points z_1 and z_2 lie on the same straight line through the origin and are on opposite sides of the origin.

It is easy to prove by induction the more general inequality

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|. \tag{9}$$

We shall only examine under which conditions equality holds in (9). Assume that $z_k \neq 0$ for $k = 1, \ldots, n$. If

$$|z_1| + \cdots + |z_n| = |z_1 + \cdots + z_n| = |(z_1 + z_2) + z_3 + \cdots + z_n|, \tag{10}$$

then

$$\begin{aligned} |z_1| + \cdots + |z_n| &\leq |z_1 + z_2| + |z_3| + \cdots + |z_n| \\ &\leq |z_1| + |z_2| + |z_3| + \cdots + |z_n|. \end{aligned}$$

This is equivalent to

$$|z_1| + |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|,$$

and hence

$$|z_1 + z_2| = |z_1| + |z_2|.$$

Thus $z_2\bar{z}_1 \geq 0$ and, since $z_1z_2 \neq 0$, we have $z_2/z_1 > 0$. Similarly, we conclude that

$$\frac{z_k}{z_j} > 0 \quad \text{for} \quad k, j = 1, \ldots, n. \tag{11}$$

Conversely, if inequalities (11) are valid, we have

$$\begin{aligned}|z_1 + \cdots + z_n| &= |z_1| \left|1 + \frac{z_2}{z_1} + \cdots + \frac{z_n}{z_1}\right| \\ &= |z_1| \left(1 + \frac{z_2}{z_1} + \cdots + \frac{z_n}{z_1}\right) \\ &= |z_1| \left(1 + \left|\frac{z_2}{z_1}\right| + \cdots + \left|\frac{z_n}{z_1}\right|\right) \\ &= |z_1| + \cdots + |z_n|.\end{aligned}$$

Accordingly, by the assumption that $z_k \neq 0$ $(k = 1, \ldots, n)$, equality (10) is true if and only if inequalities (11) are satisfied, i.e., if and only if the points $z_k \neq 0$ $(k = 1, \ldots, n)$ lie on the same ray issuing from the origin.

1.3 Monotone Functions

In what follows let f be a real-valued function defined on an interval $I \subset R$.

Definition 1. *A function f is called nondecreasing on I if for each pair of different points $x_1, x_2 \in I$, the condition*

$$(x_1 - x_2)\,(f(x_1) - f(x_2)) \geq 0$$

is valid, and increasing if the following strict inequality

$$(x_1 - x_2)\,(f(x_1) - f(x_2)) > 0$$

holds.

Definition 2. *A function f is called nonincreasing on I if, for each pair of different points $x_1, x_2 \in I$,*

$$(x_1 - x_2)\,(f(x_1) - f(x_2)) \leq 0,$$

and decreasing if

$$(x_1 - x_2)\,(f(x_1) - f(x_2)) < 0.$$

Definition 3. *If for a function f either Definition 1 or Definition 2 is valid, then f is called a monotone function.*

Clearly, if f is nondecreasing, or increasing, then $-f$ is nonincreasing, or decreasing respectively. We shall, therefore, give only the properties of nondecreasing functions, since they can be carried over to other monotone functions:

1° Let f and g be nondecreasing functions on I. Then $f + g$ have the same property.

2° If f is a nondecreasing function and if λ is a nonnegative real number, then λf is nondecreasing.

3° If f and g are nonnegative and nondecreasing functions, then fg is also a nondecreasing function.

4° If f and g are monotone functions (without further specification), it cannot be concluded that $f + g$ is monotone.

5° If f is a positive and a nondecreasing function, then $1/f$ is nonincreasing.

In practice we often use the following criterion for monotony:

Theorem 1. *If f is a differentiable function on I, it is monotone on I if and only if the sign of f' remains the same throughout I. In particular, if $f'(x) > 0$, except maybe on a set of points of I which does not contain any interval of I, then and only then f is an increasing function; if $f'(x) \geq 0$, then f is nondecreasing; if $f'(x) < 0$ on I, then f is a decreasing function and for $f'(x) \leq 0$, f is nonincreasing.*

The above statements immediately follow from the definitions of monotone functions.

The following result is of some use in Analysis:

Theorem 2. *If f is a nonnegative, nondecreasing and integrable function on $[0, a]$, then $x \mapsto \frac{1}{x}\int_0^x f(u)\,du$ is a nondecreasing function on $[0, a]$.*

A generalization of Theorem 2 is

Theorem 3. *Let f and g be respectively nonnegative nondecreasing, and nonnegative nonincreasing integrable functions on $[a, b]$. If we put $F(x) = \int_a^x f(u)\,du$, $G(x) = \int_a^x g(u)\,du$, $h(x) = F(x)/G(x)$, and $S = \{x \mid x \in [a, b], G(x) \neq 0\}$, then h is defined and nondecreasing on S.*

The above theorem, as well as the following result, is demonstrated in [1] by T. E. Mott.

Theorem 4. *If f is a nondecreasing nonnegative integrable function on $[a, b]$, then*

$$\frac{1}{x-a}\int_a^x f(u)\,du \leq \frac{1}{b-a}\int_a^b f(u)\,du \leq \frac{1}{b-x}\int_x^b f(u)\,du$$

for every $x \in (a, b)$.

For other results in the above direction see the interesting papers [2] of R. M. Redheffer and [3] of R. P. Boas.

There exist many concepts of monotony from which we mention in particular cyclic monotony defined and explored by S. N. BERNSTEIN in a multitude of papers.

Definition 4. *If two functions f and g are either both nondecreasing, increasing, nonincreasing or decreasing, we say that they are monotone in the same sense.*

By analogy with the definition of monotone functions, we define monotone sequences:

Definition 5. *For a sequence of real numbers $a = (a_1, \ldots, a_n)$ we say that it is nondecreasing if for all $k = 1, \ldots, n - 1$ we have $a_k \leq a_{k+1}$ and that it is increasing if $a_k < a_{k+1}$ for $k = 1, \ldots, n - 1$.*

Definition 6. *For a sequence of real numbers $a = (a_1, \ldots, a_n)$ we say that it is nonincreasing if for all $k = 1, \ldots, n - 1$ we have $a_k \geq a_{k+1}$ and that it is decreasing if $a_k > a_{k+1}$ for $k = 1, \ldots, n - 1$.*

Definition 7. *If for a sequence of real numbers either Definition 5 or Definition 6 is valid, we say that the sequence in question is monotone.*

In Definitions 5, 6, 7 we supposed that the sequences were finite. Similar definitions are used for infinite sequences.

Finally, we give the definition of similarly ordered sequences.

Definition 8. *Two sequences of real numbers $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are similarly ordered if and only if for each pair (i, j), where $i, j = 1, \ldots, n$, we have*

$$(a_i - a_j)(b_i - b_j) \geq 0,$$

and oppositely ordered if and only if this inequality is reversed.

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1.4 Convex Functions

1.4.1 Definitions of Jensen Convex Functions

In this section I denotes an interval (a, b), $\bar{I}$ a segment $[a, b]$, and f denotes a real function defined on I or $\bar{I}$.

Definition 1. *A function f is called convex in the* JENSEN *sense, or J-convex, on $\bar{I}$ if for every two points $x, y \in \bar{I}$ the following inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1}$$

is valid.

Definition 2. *A J-convex function f is said to be strictly J-convex on $\bar{I}$ if for every pair of points $x, y \in \bar{I}$, $x \neq y$, strict inequality holds in* (1).

Definition 3. *A function f is called J-concave (strictly J-concave) on $\bar{I}$ if the function $x \mapsto -f(x)$ is J-convex (strictly J-convex) on $\bar{I}$.*

Remark 1. Sometimes, in the literature, functions satisfying the conditions of Definition 1 are called J-nonconcave and those which are defined as J-concave are called J-nonconvex, whereas strictly J-convex and strictly J-concave functions are called J-convex and J-concave, respectively.

By analogy with J-convex functions we define convex sequences, as follows:

Definition 4. *The real sequence $a = (a_1, \ldots, a_n)$ is convex if for all $k = 2, \ldots, n-1$*

$$2a_k \leq a_{k-1} + a_{k+1}.$$

A similar definition can also be applied if the sequence is infinite.

Theorem 1. *Let f be J-convex on $\bar{I'}$ and g J-convex on $\bar{I''}$, and let $\bar{I} = \bar{I'} \cap \bar{I''}$ under the condition that I has at least two points. Then:*

1° $x \mapsto \max(f(x), g(x))$ *is J-convex on $\bar{I}$;*

2° $x \mapsto h(x) = f(x) + g(x)$ *is J-convex on $\bar{I}$;*

3° $x \mapsto f(x)\, g(x)$ *is J-convex on $\bar{I}$ provided that both f and g are positive and nondecreasing functions on $\bar{I}$;*

4° $x \mapsto h(x) = g(f(x))$ *is J-convex on $\bar{I'}$ provided that g is a nondecreasing function on $\bar{I''}$ and $[f(a), f(b)] \subset \bar{I''}$.*

Proof. The assertion 1° follows from

$$\max(x+y, u+v) \leq \max(x, u) + \max(y, v),$$

which holds for all real numbers x, y, u, v.

For $x, y \in \bar{I}$ we have

$$h\left(\frac{x+y}{2}\right) = f\left(\frac{x+y}{2}\right) + g\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \frac{g(x)+g(y)}{2} = \frac{h(x)+h(y)}{2},$$

which proves 2°.

The assumption that f and g are nondecreasing on $\bar{I}$ implies

$$\big(f(x) - f(y)\big)\big(g(y) - g(x)\big) \le 0 \qquad (x, y \in \bar{I}),$$

i.e.,

$$f(x)\,g(y) + f(y)\,g(x) \le f(x)\,g(x) + f(y)\,g(y). \tag{2}$$

If we multiply the following inequalities:

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}, \qquad g\left(\frac{x+y}{2}\right) \le \frac{g(x) + g(y)}{2},$$

where f and g are, by assumption, positive, then applying (2), we find the desired result 3°.

Since g is nondecreasing on $\overline{I''}$, from

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} \qquad (x, y \in \bar{I'})$$

we get

$$h\left(\frac{x+y}{2}\right) = g\left(f\left(\frac{x+y}{2}\right)\right) \le g\left(\frac{f(x)+f(y)}{2}\right) \le \frac{g(f(x)) + g(f(y))}{2} = \frac{h(x) + h(y)}{2},$$

which proves 4°.

Next, we prove an interesting inequality for J-convex functions which is, in some sense, the best of its kind.

Theorem 2. *Suppose that f is J-convex on $\bar{I}$. For any points $x_1, \ldots, x_n \in \bar{I}$ and any rational nonnegative numbers $r_1, \ldots, r_n$ such that $r_1 + \cdots + r_n = 1$, we have*

$$f\left(\sum_{i=1}^{n} r_i x_i\right) \le \sum_{i=1}^{n} r_i f(x_i). \tag{3}$$

Proof. *Case 1*: $r_i = \frac{1}{n}$ $(i = 1, \ldots, n)$. In that case (3) becomes

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{4}$$

The proof of (4) is by induction on n. For $n = 2$, (4) holds because it agrees with Definition 1. Suppose that Theorem 2 holds for $n = 2^k$, where k is a natural number. For $x_1, \ldots, x_m \in \bar{I}$ and for $m = 2^{k+1} = 2 \cdot 2^k = 2n$ we have

$$f\left(\frac{x_1 + \cdots + x_m}{m}\right) = f\left(\frac{\frac{1}{n}\sum_{m=1}^{n} x_m + \frac{1}{n}\sum_{m=1}^{n} x_{m+n}}{2}\right)$$

$$\le \frac{f\left(\frac{1}{n}\sum_{m=1}^{n} x_m\right) + f\left(\frac{1}{n}\sum_{m=1}^{n} x_{m+n}\right)}{2} \le \frac{\sum_{m=1}^{n} f(x_m) + \sum_{m=1}^{n} f(x_{m+n})}{2n} = \frac{\sum_{m=1}^{2n} f(x_m)}{2n}.$$

Thus (4) holds for every natural number $n \in \{2, 2^2, 2^3, \ldots\}$.

Let us prove that if (4) holds for $n > 2$, then it also holds for $n - 1$. Let $x_1, \ldots, x_{n-1} \in \bar{I}$. For numbers $x_1, \ldots, x_{n-1}$ and $x_n = \frac{1}{n-1}(x_1 + \cdots + x_{n-1})$ (4) holds, i.e.,

$$f\left(\frac{x_1 + \cdots + x_{n-1} + \dfrac{x_1 + \cdots + x_{n-1}}{n-1}}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_{n-1}) + f\left(\dfrac{x_1 + \cdots + x_{n-1}}{n-1}\right)}{n}. \tag{5}$$

The left-hand side of this inequality, after rearranging, becomes

$$f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right),$$

so that (5) reads

$$f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right) \leq \frac{1}{n}\left(f(x_1) + \cdots + f(x_{n-1})\right) + \frac{1}{n} f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right),$$

which yields

$$f\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right) \leq \frac{f(x_1) + \cdots + f(x_{n-1})}{n-1}.$$

Therefore, if (4) is true for $n > 2$, it is also true for $n - 1$.

This completes the proof of Theorem 2 for the above case.

Case 2: Since $r_1, \ldots, r_n$ are nonnegative rational numbers there is a natural number m and nonnegative integers $p_1, \ldots, p_n$ such that $m = p_1 + \cdots + p_n$ and $r_i = \frac{p_i}{m}$ $(i = 1, \ldots, n)$. Now, by Case 1, we have

$$f\left(\frac{(x_1 + \cdots + x_1) + \cdots + (x_n + \cdots + x_n)}{m}\right) \leq \frac{(f(x_1) + \cdots + f(x_1)) + \cdots + (f(x_n) + \cdots + f(x_n))}{m}, \tag{6}$$

where in the first bracket there are p_1 terms, ..., in the n-th bracket p_n terms. Thus (6) reads

$$f\left(\frac{1}{m} \sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} \frac{p_i}{m} f(x_i).$$

Remark 2. The definition of a J-convex function and Theorems 1 and 2 apply for any real valued function f defined on a subset $\bar{I}$ of an n-dimensional Euclidean space, provided that $\bar{I}$ has the mid-point property, i.e., $x, y \in \bar{I}$ implies $\frac{x+y}{2} \in \bar{I}$.

J. L. W. V. Jensen (see [1] and [2]) was the first to define convex functions using inequality (1) and to draw attention to their importance. He also proved Theorem 2. His impression, which we quote here, was completely justified: "Il me semble que la notion de fonction convexe est à peu près aussi fondamentale que celles-ci: fonction positive, fonction croissante. Si je ne me trompe pas en ceci, la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles" (see [2], p. 191).

However, even before JENSEN, there were results which refer to convex functions. So, for example, in 1889 O. HÖLDER [3] proved inequality (3) under the condition that f is twice differentiable on I and that $f''(x) \geq 0$ (i.e. f is convex, though this was not explicitly specified in the article). Let us also mention a result from 1893 due to O. STOLZ [4] on the existence of left and right derivatives of a continuous function which satisfies (1). This result is formulated in Theorem 1 in 1.4.4. We agree with T. POPOVICIU (see [5], p. 48) that it seems that STOLZ first introduced convex functions by proving the result just referred to. A result of J. HADAMARD [6] from 1893 also belongs to the period which precedes the publication of JENSEN's articles: If a function f is differentiable, and if its derivative is an increasing function on $\bar{I}$, then for all $x_1, x_2 \in I$ $(x_1 \neq x_2)$,

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\, dx$$

(see also p. 441 of [7]).

1.4.2 Continuity of Jensen Convex Functions

If f is a J-convex function on $\bar{I}$, then, by Theorem 2 in 1.4.1,

$$f(\lambda x + (1 - \lambda)\, y) \leq \lambda f(x) + (1 - \lambda)\, f(y) \tag{1}$$

holds for all $x, y \in \bar{I}$ and for all rational numbers $\lambda \in [0, 1]$. Hence if f is continuous, then (1) holds for all real numbers $\lambda \in [0, 1]$.

However, a J-convex function f on $\bar{I}$ is not necessarily continuous on $\bar{I}$. We give an example of such a function (see [7]). Let f be any discontinuous solution of CAUCHY's functional equation

$$f(x + y) = f(x) + f(y)$$

(the existence of such solutions was proved by G. HAMEL [8]; for the general solution of this functional equation see J. ACZÉL [9]). The function g defined by $g(x) = \max(x^2, f(x) + x^2)$ is J-convex and discontinuous.

Furthermore, the function f defined by

$$f(x) = x \quad (-1 < x < 1), \quad f(-1) = f(1) = 2$$

is J-convex on $[-1, +1]$, continuous on $(-1, 1)$ but it is not continuous at the end points of the interval.

There are many results which under various conditions on a J-convex function guarantee its continuity. We mention only some of them. One of the most important results is obtained by use of (1) and it is (see Jensen [1], and F. Bernstein and G. Doetsch [10]):

Theorem 1. *If a J-convex function is defined and bounded from above on I, then it is continuous on I.*

This result was generalized by W. Sierpiński [11] who proved that a J-convex function f which is bounded from above by a measurable function (in the Lebesgue sense) is continuous. A. Ostrowski [12] extended that result by proving that a J-convex function which is bounded on a set of positive measure is also continuous. S. Kurepa [13] observed that a J-convex function f is bounded on a mid-point set $\frac{1}{2}(T + T) = \left\{\frac{t+s}{2} \,\middle|\, t, s \in T\right\}$ if it is bounded on T. Hence if a set T is such that the interior measure of $\frac{1}{2}(T + T)$ is strictly positive then boundedness of f on T implies its continuity; if T is of positive measure so is $\frac{1}{2}(T + T)$. An interesting fact in that connection is that there are sets of measure zero for which $T + T$ is an interval. For some further results in this direction see R. Ger and M. Kuczma [14].

1.4.3 Convex Functions

Definition 1. *A function f is called convex on a segment $\bar{I}$ if and only if*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{1}$$

holds for all $x, y \in \bar{I}$ and all real numbers $\lambda \in [0, 1]$.

A convex function f on $\bar{I}$ is said to be strictly convex if strict inequality holds in (1) for $x \neq y$.

Obviously a convex function is J-convex and every continuous J-convex function is convex.

Remark 1. A function f is convex on $\bar{I}$ if and only if

$$f\left(\frac{px + qy}{p + q}\right) \leq \frac{pf(x) + qf(y)}{p + q} \tag{2}$$

holds for all $x, y \in \bar{I}$ and all real numbers $p, q > 0$.

The geometric meaning of strict convexity of a function f on $\bar{I}$ is that the line segment on I joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f.

Since this geometric interpretation replaces the analytic condition of convexity it is often taken as a definition of a convex function (see N. Bourbaki [15]).

Theorem 1. *A function $f\colon \bar{I} \to R$ is convex on $\bar{I}$ if and only if for any three points x_1, x_2, x_3 $(x_1 < x_2 < x_3)$ from $\bar{I}$ the following inequality holds:*

$$(3)\quad \begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0.$$

Proof. Putting in (1) $x = x_1$, $y = x_3$, $\lambda x + (1 - \lambda) y = x_2$, after rearranging, we get (3). Conversely, putting in (3) $x_1 = x$, $x_2 = \lambda x + (1 - \lambda) y$, $x_3 = y$, we get (1) with the condition $x < y$. If $x > y$, then putting $x_1 = y$, $x_2 = \lambda x + (1 - \lambda) y$, $x_3 = x$, (1) is obtained again.

Consider $x_1, x_2, x_3 \in \bar{I}$, where $x_1 < x_2 < x_3$, and the values $f(x_1)$, $f(x_2)$, $f(x_3)$. The area P of the triangle whose vertices are $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$ is given by

$$P = \frac{1}{2} \begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix}.$$

The function whose graph is represented in Fig. 1 is convex: in that case $P > 0$. In Fig. 2 we have $P < 0$ for the case of a concave function.

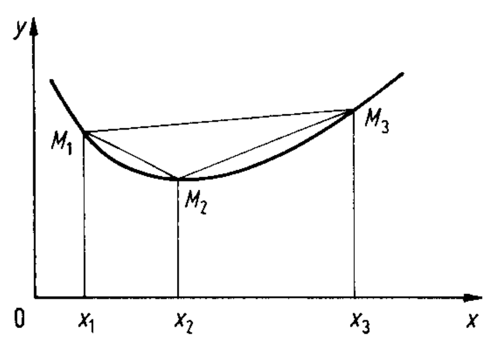

Fig. 1

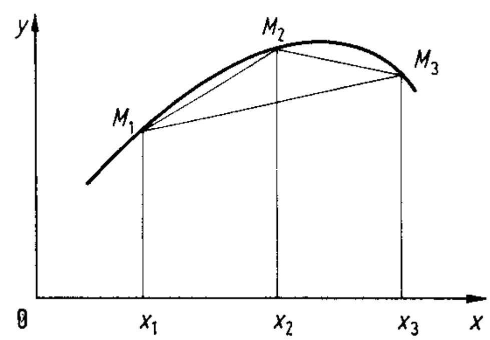

Fig. 2

Remark 2. By rewriting (3), we find that f is convex if and only if

$$(4)\quad \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

Inequality (4) is often written in the form

$$(5)\quad [x_1, x_2, x_3; f] \geq 0,$$

where in general $[x_1, x_2, \ldots, x_n; f]$ is defined by the recursive relation

$$[x_1, \ldots, x_n; f] = \frac{[x_2, \ldots, x_n; f] - [x_1, \ldots, x_{n-1}; f]}{x_n - x_1};$$

$$[x; f] = f(x).$$

L. Galvani [16] was the first to define convex functions by inequality (5).

Definition 2. *A function $f\colon \bar{I} \to R$ is called convex on $\bar{I}$ of order $n \geq 2$ if and only if, for all $x_1, \ldots, x_{n+2} \in \bar{I}$,*

$$[x_1, \ldots, x_{n+2}; f] \geq 0.$$

For convex function of order n results similar to those listed about the continuity of convex functions have also been obtained (see [5], [17] and [59]).

1.4.4 Continuity and Differentiability of Convex Functions

O. Stolz [4] has proved the following result: If f is a continuous function on $\bar{I}$ and if f satisfies

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2},$$

then f has left and right derivatives at each point of I.

Theorem 1. *Suppose that f is a convex function on $\bar{I}$. Then: 1° f is continuous on I; 2° f has left and right derivatives on I and 3° $f'_-(x) \leq f'_+(x)$ for each $x \in I$.*

Proof. We shall give a geometric proof of the first part of this theorem (following W. Rudin: Real and complex analysis. New York 1966, pp. 60—61). Suppose that $a < x_1 < x_2 < x_3 < x_4 < b$ and denote the point $(x_i, f(x_i))$ by X_i ($i = 1, 2, 3, 4$). Then X_2 is either on or below the line which connects X_1 and X_3, while X_3 is either on or above the line which joins X_1 and X_2. Furthermore, X_3 is either on or below the line which joins X_2 and X_4. When $x_3 \to x_2$, we have $X_3 \to X_2$, i.e. $f(x_3) \to f(x_2)$. Applying the same procedure to the left-hand limit, the continuity of f follows.

Notice that this theorem states the fact that a function convex on $\bar{I}$ is continuous only on I and need not be continuous on $\bar{I}$. Indeed, the function $f(x) = 0$ $(0 \leq x < 1)$ and $f(1) = 1$ is convex on $[0, 1]$ but it is not continuous on that segment.

The following theorem provides a criterion for convexity:

Theorem 2. *If the function f has a second derivative in the interval I, then*

$$f''(x) \geq 0 \ \text{for}\ x \in I$$

is a necessary and sufficient condition for the function f to be convex on that interval.

Proof. The condition is necessary. Indeed if the function f is convex, then for any three distinct points $x_1, x_2, x_3 \in I$ inequality (3) in 1.4.3 holds. This inequality can be written in the form

$$\frac{\dfrac{f(x_3)-f(x_2)}{x_3-x_2}-\dfrac{f(x_2)-f(x_1)}{x_2-x_1}}{x_3-x_1}\geq 0,$$

which implies $f''(x_3)\geq 0$ for all $x_3\in I$.

The condition is sufficient. Let x_1 and x_2 $(x_1<x_2)$ be two arbitrary points of the interval (a,b). Applying TAYLOR's formula in a neighbourhood of the point $\frac{1}{2}(x_1+x_2)$, we get

$$f(x_1)=f\left(\frac{x_1+x_2}{2}\right)+\left(x_1-\frac{x_1+x_2}{2}\right)f'\left(\frac{x_1+x_2}{2}\right)+\frac{1}{2}\left(x_1-\frac{x_1+x_2}{2}\right)^2 f''(\xi_1),$$

$$f(x_2)=f\left(\frac{x_1+x_2}{2}\right)+\left(x_2-\frac{x_1+x_2}{2}\right)f'\left(\frac{x_1+x_2}{2}\right)+\frac{1}{2}\left(x_2-\frac{x_1+x_2}{2}\right)^2 f''(\xi_2),$$

where $\xi_1\in\left(x_1,\frac{x_1+x_2}{2}\right)$ and $\xi_2\in\left(\frac{x_1+x_2}{2},x_2\right)$.

From there we get

$$\frac{f(x_1)+f(x_2)}{2}=f\left(\frac{x_1+x_2}{2}\right)+\frac{1}{16}(x_2-x_1)^2\,[f''(\xi_1)+f''(\xi_2)]. \tag{1}$$

Since $f''(\xi_1)\geq 0$ and $f''(\xi_2)\geq 0$, from (1) we get

$$\frac{f(x_1)+f(x_2)}{2}\geq f\left(\frac{x_1+x_2}{2}\right)\qquad (x_1,x_2\in I)$$

which means that the function f is J-convex. Since the second derivative exists, f is continuous on I, and therefore, J-convexity implies convexity as defined by Definition 1 of 1.4.3.

This completes the proof.

Similarly it can be shown: If $f''(x)\leq 0$ on I, the function f is concave on I.

The following, more general, theorem also holds:

Theorem 3. 1° *A function f is convex on $\bar{I}$ if and only if for every point $x_0\in\bar{I}$ the function $x\mapsto\frac{f(x)-f(x_0)}{x-x_0}$ is nondecreasing on $\bar{I}$.*

2° *A differentiable function f is convex if and only if f' is a nondecreasing function on $\bar{I}$.*

3° *A twice differentiable function f is convex on $\bar{I}$ if and only if $f''(x)\geq 0$ for all $x\in I$.*

1.4.5 Logarithmically Convex Functions

Definition 1. *A function f is logarithmically convex on $\bar{I}$, if the function f is positive and the function $x\mapsto\log f(x)$ is convex on $\bar{I}$.*

Theorem 1. *If f_1 and f_2 are logarithmically convex functions on $\bar{I}$, then the functions $x \mapsto f_1(x) + f_2(x)$ and $x \mapsto f_1(x) f_2(x)$ are also logarithmically convex on $\bar{I}$.*

P. MONTEL [18] has proved the following theorem:

Theorem 2. *A positive function f is logarithmically convex if and only if $x \mapsto e^{ax} f(x)$ is a convex function for all real values of a.*

For a development of the results of P. MONTEL see particularly the papers [19] and [20] of G. VALIRON.

Theorem 3 (Three lines theorem). *Let f be a complex-valued and an analytic function of a complex variable $z = x + iy$. Suppose that f is defined and bounded in the strip $a \le x \le b$, $-\infty < y < +\infty$. Let $M\colon [a, b] \to R$ be defined by*

$$M(x) = \sup_{-\infty < y < +\infty} |f(x + iy)|.$$

Then M is logarithmically convex on $[a, b]$.

This theorem is due to M. RIESZ.

Theorem 4 (Three circles theorem). *Let f be a complex-valued and an analytic function in the annulus domain $a < |z| < b$. Then $\log M(r)$ is a convex function of $\log r$ $(a < r < b)$, where*

$$M(r) = \max_{|z| = r} |f(z)|.$$

This result is due to J. HADAMARD.

Theorem 5. *Let f be a complex-valued and an analytic function in the annulus domain $a < |z| < b$. Let $1 \le p < +\infty$. Then $\log M_p(r)$ is a convex function of $\log r$ $(a < r < b)$, where*

$$M_p(r) = \left(\int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi \right)^{\frac{1}{p}}.$$

This theorem is due to G. H. HARDY.

Theorem 6. *Let f be a complex-valued and an analytic function in the annulus domain $a < |z| < b$. Let $0 < \alpha < 1$ and*

$$M_\alpha(r) = \max_{|z_1| = |z_2| = r} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}.$$

Then $M_\alpha(r)$ is a convex function of $\log r$ $(a < r < b)$.

Concerning Theorems 3—6 and their generalizations, see [21], [22] and [23].

1.4.6 Some Extensions of the Concept of Convex Functions

We have already given the definition of a convex function of order n, and now we shall quote some results which extend the concept of a convex function.

I. E. OVČARENKO [24] introduced the following definition:

Definition 1. *A bounded function f is convex on $\bar{I}$ with respect to the function g if for every $x \in I$ there is a number $\delta_x > 0$ such that*

$$f(x_1)\, g(x - x_3) + f(x)\, g(x_3 - x_1) + f(x_3)\, g(x_1 - x) \leq 0$$

for $x_1 < x < x_3$ and $x_3 - x_1 < \delta_x$.

For $g(x) = x$, this definition reduces to the definition of convex functions.

In the book [25] of T. BONNESEN we find:

Definition 2. *Let $g\colon R^2 \to R$ be a continuous function on the square $(a, b) \times (a, b)$ and let $\alpha \leq g(x_1, x_2) \leq 1 - \alpha$ where α $(0 < \alpha < 1)$ is a fixed number. A function f is convexoidal on $[a, b]$ if for all $x_1 < x_2$,*

$$f\big(x_1 g(x_1, x_2) + x_2(1 - g(x_1, x_2))\big) \leq g(x_1, x_2)\, f(x_1) + \big(1 - g(x_1, x_2)\big)\, f(x_2).$$

For $g(x_1, x_2) = 1/2$ this definition yields the definition of J-convex functions.

The following theorem holds for convex functions (see, for example, M. A. KRASNOSEL'SKIĬ and YA. B. RUTICKIĬ [26]):

Theorem 1. *Any convex function f such that $f(a) = 0$ can be represented in the form*

$$f(u) = \int_a^u p(t)\, dt,$$

where p is a nondecreasing right-continuous function.

The following definition is connected to the above theorem.

Definition 3. *A function f is called an N-function if it can be represented in the form*

$$f(u) = \int_0^{|u|} p(t)\, dt,$$

where the function p is continuous from the right for $t \geq 0$, positive for $t > 0$, and nondecreasing, and such that

$$p(0) = 0, \qquad \lim_{t \to +\infty} p(t) = +\infty.$$

Definition 4. *Let p be a function with the same properties as in Definition 3. For $s \geq 0$, define the function q by*

$$q(s) = \sup_{p(t) \leq s} t.$$

Then functions

$$f(u) = \int_0^{|u|} p(t)\, dt \quad \text{and} \quad g(v) = \int_0^{|v|} q(s)\, ds$$

are called mutually complementary N-functions.

Remark. If p is a continuous and monotone increasing function, then q is the inverse function of p. Otherwise, q is called the right-inverse function of p. It can easily be verified that q has the same properties as those given by Definition 3 for the function p.

More about N-functions, complementary N-functions and their applications to the ORLICZ spaces can be found in [26].

We finally mention a definition of intern functions due to Á. CSÁSZÁR [27]:

Definition 5. *A real function f defined on I is called an intern function on I if for all $x, y \in I$,*

$$\min\big(f(x), f(y)\big) \leq f\Big(\frac{x+y}{2}\Big) \leq \max\big(f(x), f(y)\big).$$

Properties of intern functions were the subject of study of Á. CSÁSZÁR (see [27] and [28]) and S. MARCUS (see [29] and [30]).

In terms of normal mean values, J. ACZÉL [31] gave a generalization of the notion of convex function.

For an interesting and important extension of the idea of convexity, see the paper [32] of A. OSTROWSKI.

The theory of J-convex functions as opposed to the theory of convex functions, was only partially transposed to the functions of several variables. Some properties of these functions which correspond to those cited by F. BERNSTEIN and G. DOETSCH [10], were given by H. BLUMBERG [33] and E. MOHR [34]. S. MARCUS [35] gave an analogue of the theorem of A. OSTROWSKI [12] and of M. HUKUHARA [36] for the functions of several variables.

As a natural generalization of the concept of convexity for functions of two real variables, the concepts of subharmonic and of double convex functions have been studied. Concerning such functions, see papers of P. MONTEL [21] and M. NICOLESCO [37].

A great deal of intensive work has been done recently on the theory of convex and generalized convex functions. Besides the literature quoted in the books of E. F. BECKENBACH and R. BELLMAN [38], G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA [39], and T. POPOVICIU [5] and in the paper [7] of E. F. BECKENBACH, also see [40]—[64].

Concerning the applications of the convexity of order n, see, in particular, the expository article [65] of E. MOLDOVAN-POPOVICIU.

1.4.7 Hierarchy of Convexity

Let $K(b)$ be the class of all functions $f\colon R \to R$ which are continuous and nonnegative on the segment $I = [0, b]$ and such that $f(0) = 0$.

The mean function F of the function $f \in K(b)$, defined by

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad (0 < x \leq b), \quad F(0) = 0,$$

also belongs to the class $K(b)$.

Let $K_1(b)$ denote the class of functions $f \in K(b)$ convex on I.

Let $K_2(b)$ denote the class of functions $f \in K(b)$ for which $F \in K_1(b)$.

Let $K_3(b)$ denote the class of functions f which are starshaped with respect to the origin on the segment I, i.e., the class of those functions f with the property that for all $x \in I$ and all t $(0 \leq t \leq 1)$ the following inequality holds:

$$f(tx) \leq t f(x).$$

We say that a function f belongs to the class $K_4(b)$ if and only if it is superadditive on I, i.e., if and only if

$$f(x+y) \geq f(x) + f(y) \text{ for } x, y \text{ and } x+y \text{ in } I.$$

If F belongs to the class $K_3(b)$, we say that f belongs to $K_5(b)$, and if F belongs to $K_4(b)$, we say that f is from $K_6(b)$

A. M. BRUCKNER and E. OSTROW [55] have proved that the following inclusions hold:

$$K_1(b) \subset K_2(b) \subset K_3(b) \subset K_4(b) \subset K_5(b) \subset K_6(b).$$

E. F. BECKENBACH [66] has given examples which show that

$$K_6(b) \neq K_5(b),\ K_5(b) \neq K_4(b),\ K_4(b) \neq K_3(b)\ ,\ K_3(b) \neq K_2(b),$$
$$K_2(b) \neq K_1(b).$$

The following result of M. PETROVIĆ [67] is also related to this:

Theorem 1. *If f is a convex function on the segment $\bar{I} = [0, a]$, if $x_i \in \bar{I}$ $(i = 1, \ldots, n)$ and $x_1 + \cdots + x_n \in \bar{I}$, then*

$$(1) \qquad f(x_1) + \cdots + f(x_n) \leq f(x_1 + \cdots + x_n) + (n-1) f(0).$$

Proof. Putting $p = x_1$, $q = x_2$, $x = x_1 + x_2$, $y = 0$ $(x_1, x_2 > 0)$ in (2) of 1.4.3 we get

$$f(x_1) \leq \frac{x_1 f(x_1 + x_2)}{x_1 + x_2} + \frac{x_2}{x_1 + x_2} f(0). \tag{2}$$

Interchanging x_1 and x_2 we find

$$f(x_2) \leq \frac{x_2 f(x_1 + x_2)}{x_1 + x_2} + \frac{x_1}{x_1 + x_2} f(0). \tag{3}$$

Adding (2) and (3) we get

$$f(x_1) + f(x_2) \leq f(x_1 + x_2) + f(0). \tag{4}$$

Theorem 1 is therefore true for $n = 2$. Suppose that it holds for some n. Then, by (4) we have

$$\begin{aligned} f(x_1 + \cdots + x_n + x_{n+1}) &= f\big((x_1 + \cdots + x_n) + x_{n+1}\big) \\ &\geq f(x_1 + \cdots + x_n) + f(x_{n+1}) - f(0), \end{aligned}$$

and by the induction hypothesis,

$$f(x_1) + \cdots + f(x_n) + f(x_{n+1}) \leq f(x_1 + \cdots + x_{n+1}) + nf(0).$$

This completes the inductive proof.

For $n = 2$, we get that if $f(0) = 0$ and if f is convex, then f is superadditive.

Some generalizations of (1) were given by D. Marković [68], P. M. Vasić [69], and J. D. Kečkić and I. B. Lacković (see 3.9.57).

Finally, it should be emphasized that the theory of convexity developed in this section, taken together with a few elementary devices, can be used to derive a large number of the most familiar and important inequalities of Analysis.

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2. General Inequalities

2.1 Fundamental Inequalities

2.1.1 Simple Means

Definition 1. *Let $a = (a_1, \ldots, a_n)$ be a given sequence of positive numbers. Then the harmonic mean $H_n(a)$ of the numbers $a_1, \ldots, a_n$ is defined as*

$$H_n(a) = \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}};$$

their geometric mean $G_n(a)$ is defined as

$$G_n(a) = (a_1 \cdots a_n)^{1/n};$$

and their arithmetic mean $A_n(a)$ is defined as

$$A_n(a) = \frac{a_1 + \cdots + a_n}{n}.$$

Theorem 1. *For any finite sequence of positive numbers $a = (a_1, \ldots, a_n)$ we have*

$$\text{(1)} \quad \min(a_1, \ldots, a_n) \leq H_n(a) \leq G_n(a) \leq A_n(a) \leq \max(a_1, \ldots, a_n),$$

with equality if and only if

$$a_1 = \cdots = a_n.$$

For the sake of brevity, the inequality between the harmonic and geometric means will be called inequality (HG), while the inequality between the geometric and arithmetic means will be called inequality (GA).

There is a large number of proofs of Theorem 1 in mathematical literature, especially proofs of inequality (GA). The most complete information, so far, can be found in the book [1], and a number of proofs can be found in the books [2] and [3].

The first concept of the arithmetic and geometric means of two real positive numbers is probably due to the Pythagoreans. It is likely that they knew of the inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \quad (a, b > 0),$$

but there is no doubt that it was proved by EUCLID.

The first, and one of the most beautiful proofs of inequality (GA), was certainly the one given by A. CAUCHY (see [4]; his original proof can be found in [1], pp. 21–22, and [3], pp. 17–18). In his proof, A. CAUCHY was the first to use the method of regressive induction. However there is a shortcoming in this proof in that the case of equality was not discussed.

We shall give here another beautiful proof due to J. LIOUVILLE [5] which is, as far as we know, not quoted in the literature on this subject. This must be the reason why this proof has been rediscovered many times later (see [1], pp. 28–29). A translation from French of that proof runs:

We shall prove that

$$\frac{x_1 + \cdots + x_n}{n} = \sqrt[n]{x_1 \cdots x_n} \tag{2}$$

whenever all the quantities $x_1, x_2, \ldots, x_n$ are equal, and that

$$\frac{x_1 + \cdots + x_n}{n} > \sqrt[n]{x_1 \cdots x_n} \tag{3}$$

in other cases.

Notice that formulas (2) and (3) are true for $n = 2$, since in this case the first term is equal to

$$\frac{x_1 + x_2}{2} = \frac{(\sqrt{x_1} - \sqrt{x_2})^2}{2} + \sqrt{x_1 x_2},$$

while the second reduces to $\sqrt{x_1 x_2}$. Therefore, it is enough to show that if the formulas (2) and (3) are true for a certain value of n, they do not cease to be true if this value is increased by unity: in other words it is enough to prove that, assuming that (2) and (3) hold for a given value of n, the function

$$y = \left(\frac{x_1 + x_2 + \cdots + x_n + x_{n+1}}{n+1}\right)^{n+1} - x_1 x_2 \cdots x_n x_{n+1}$$

is always positive or zero, the later case only when $x_1 = x_2 = \cdots = x_n = x_{n+1}$. We shall treat x_{n+1} as a continuous variable, and taking the derivative of y with respect to that variable, we find that it is equal to

$$\left(\frac{x_1 + x_2 + \cdots + x_n + x_{n+1}}{n+1}\right)^{n} - x_1 x_2 \cdots x_n.$$

This derivative is therefore an increasing function of x_{n+1}, which vanishes when

$$x_{n+1} = -(x_1 + x_2 + \cdots + x_n) + (n+1)\sqrt[n]{x_1 x_2 \cdots x_n}.$$

This value reduces to x_1, when $x_1 = x_2 = \cdots = x_n$. For the values x_{n+1} which are smaller or greater than the one written, the above derivative is respectively negative or positive; therefore, the function y is respectively decreasing or increasing: it has a minimum when its derivative is zero: this minimum is

$$n x_1 x_2 \cdots x_n \left(\frac{x_1 + x_2 + \cdots + x_n}{n} - \sqrt[n]{x_1 x_2 \cdots x_n} \right),$$

and by formulas (2) and (3) it is either zero or positive: therefore, a fortiori, the function y is also ≥ 0; in order that it should reduce to zero it is necessary: 1° that the determined minimum be zero which happens when $x_1 = x_2 = \cdots = x_n$; 2° that x_{n+1} takes the value which corresponds to that minimum, that is to say, that x_{n+1} is also equal to x_1.

This completes the proof of J. Liouville.

For the numbers $a_1^{-1}, \ldots, a_n^{-1}$, by (GA) we have

$$\left(\frac{1}{a_1} \cdots \frac{1}{a_n} \right)^{\frac{1}{n}} \leq \frac{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}{n}. \tag{4}$$

Equality holds here if and only if $a_1^{-1} = \cdots = a_n^{-1}$, i.e., $a_1 = \cdots = a_n$. From (4) we get

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq (a_1 \cdots a_n)^{\frac{1}{n}};$$

that is to say, inequality (HG).

Let us now prove that

$$\min(a_1, \ldots, a_n) \leq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}. \tag{5}$$

Without loss of generality, we can suppose that

$$0 < a_1 \leq \cdots \leq a_n. \tag{6}$$

Then,

$$\min(a_1, \ldots, a_n) = a_1.$$

Using inequality (6), inequality (5) becomes

$$\frac{a_1}{a_1} + \cdots + \frac{a_1}{a_n} \leq n,$$

which is true, since by (6), we have

$$\frac{a_1}{a_k} \leq 1 \quad (k = 1, \ldots, n).$$

The inequality

$$\frac{a_1 + \cdots + a_n}{n} \leq \max(a_1, \ldots, a_n)$$

can be proved in the same way.

This completes the proof of Theorem 1.

A generalization of $G_n(a) \leq A_n(a)$ in the case $n = 2$ is given by

Theorem 2. *If $x \geq 0$, $y \geq 0$ and $1/p + 1/q = 1$ with $p > 1$, then*

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \tag{7}$$

with equality holding if and only if $x = y$.

The proof of (7) reduces, for instance, to finding the extremum of the function f defined by

$$f(x) = x^{\frac{1}{p}} y^{\frac{1}{q}} - \frac{x}{p} - \frac{y}{q}.$$

Inequality (7) is, in fact, the inequality between the weighted arithmetic and geometric means of nonnegative real numbers x and y with weights $1/p$ and $1/q$, where $1/p + 1/q = 1$ and $p > 1$ (see also 2.14).

Replacing x by x^p and y by y^q in (7), we have

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \qquad \left(x \geq 0, y \geq 0;\ \frac{1}{p} + \frac{1}{q} = 1, p > 1\right). \tag{8}$$

This inequality plays an important role in the Theory of Inequalities.

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2.1.2 Cauchy's Inequality

Theorem 1. *If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are sequences of real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \tag{1}$$

with equality if and only if the sequences a and b are proportional.

Inequality (1) is called CAUCHY's inequality, or the CAUCHY-SCHWARZ inequality, or the CAUCHY-SCHWARZ-BUNIAKOWSKI inequality. We adopt the first name.

Proof 1. Consider the quadratic polynomial in x

$$\sum_{k=1}^{n}(a_k x + b_k)^2, \tag{2}$$

i.e.,

$$\left(\sum_{k=1}^{n}a_k^2\right)x^2 + 2\left(\sum_{k=1}^{n}a_k b_k\right)x + \sum_{k=1}^{n}b_k^2. \tag{3}$$

Using (2), we conclude that (3) is nonnegative for all real x, which implies inequality (1).

Equality holds in (1) if and only if the sequences a and b are proportional as can be seen from (2).

This proves Theorem 1.

Proof 2. Since, for arbitrary real numbers x and y,

$$xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

we have, for the terms of real sequences a and b,

$$|a_k b_k| = \lambda |a_k| \frac{1}{\lambda} |b_k| \leq \frac{1}{2}\lambda^2 a_k^2 + \frac{1}{2}\frac{1}{\lambda^2}b_k^2,$$

where $\lambda \neq 0$ is an arbitrary real number.

Summing the above inequalities from $k = 1$ to $k = n$, we have

$$\sum_{k=1}^{n}|a_k b_k| \leq \frac{1}{2}\lambda^2\sum_{k=1}^{n}a_k^2 + \frac{1}{2}\frac{1}{\lambda^2}\sum_{k=1}^{n}b_k^2.$$

Choosing λ so that

$$\lambda^2\sum_{k=1}^{n}a_k^2 = \frac{1}{\lambda^2}\sum_{k=1}^{n}b_k^2 = \left(\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_k^2\right)^{1/2},$$

we obtain

$$\sum_{k=1}^{n}|a_k b_k| \leq \left(\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_k^2\right)^{1/2}. \tag{4}$$

Since

$$\left|\sum_{k=1}^{n}a_k b_k\right| \leq \sum_{k=1}^{n}|a_k b_k|,$$

from (4), we derive (1).

Theorem 2. *If the sequences a and b are complex, inequality* (1) *then reads*:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right), \tag{5}$$

with equality holding if and only if the sequences a and $\bar{b}$ are proportional.

Proof. Let λ be a complex number. Start with the identity

$$\begin{aligned}\sum_{k=1}^{n} |a_k - \lambda \bar{b}_k|^2 &= \sum_{k=1}^{n} (a_k - \lambda \bar{b}_k)\,(\bar{a}_k - \bar{\lambda} b_k) \\ &= \sum_{k=1}^{n} |a_k|^2 + |\lambda|^2 \sum_{k=1}^{n} |b_k|^2 - 2\,\mathrm{Re}\left(\bar{\lambda} \sum_{k=1}^{n} a_k b_k\right).\end{aligned}$$

If $\lambda = \left(\sum_{k=1}^{n} a_k b_k\right)\Big/\left(\sum_{k=1}^{n} |b_k|^2\right)$, where $b \neq 0$, we have

$$\sum_{k=1}^{n} |a_k - \lambda \bar{b}_k|^2 = \sum_{k=1}^{n} |a_k|^2 - \frac{\left|\sum_{k=1}^{n} a_k b_k\right|^2}{\sum_{k=1}^{n} |b_k|^2} \geq 0, \tag{6}$$

which implies CAUCHY's inequality (5) in the complex form.

In virtue of (6) we conclude that equality holds in (5) if and only if

$$a_k - \lambda \bar{b}_k = 0 \qquad (k = 1, \ldots, n),$$

i.e., if and only if the sequences a and $\bar{b}$ are proportional.

Remark. In 2.6 we shall give other proofs of Theorems 1 and 2, as well as some generalizations.

2.2 Abel's Inequality

Theorem 1. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ $(b_1 \geq \cdots \geq b_n \geq 0)$ be two sequences of real numbers, and let*

$$s_k = a_1 + \cdots + a_k \qquad (k = 1, \ldots, n).$$

If $m = \min_{1 \leq k \leq n} s_k$ and $M = \max_{1 \leq k \leq n} s_k$, then

$$m b_1 \leq a_1 b_1 + \cdots + a_n b_n \leq M b_1. \tag{1}$$

This inequality is known as ABEL's inequality.

Proof. The sum $a_1 b_1 + \cdots + a_n b_n$ can be written as follows:

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= s_1 b_1 + (s_2 - s_1)\, b_2 + \cdots + (s_n - s_{n-1})\, b_n \\ &= s_1 (b_1 - b_2) + \cdots + s_{n-1} (b_{n-1} - b_n) + s_n b_n.\end{aligned}$$

Since

$$m(b_1 - b_2) \leq s_1(b_1 - b_2) \leq M(b_1 - b_2),$$
$$\vdots$$
$$m(b_{n-1} - b_n) \leq s_{n-1}(b_{n-1} - b_n) \leq M(b_{n-1} - b_n),$$
$$mb_n \leq s_n b_n \leq M b_n,$$

by addition we get inequality (1).

2.3 Jordan's Inequality

Integrating both sides of the inequality

$$\sec^2 \theta \geq 1 \quad (0 \leq \theta < \pi/2)$$

over the interval $(0, \theta)$, we get

$$\tan \theta \geq \theta \quad (0 \leq \theta < \pi/2).$$

Therefore we obtain

$$\frac{d}{d\theta}\left(\frac{\sin \theta}{\theta}\right) = \frac{\cos \theta}{\theta^2}(\theta - \tan \theta) \leq 0 \qquad (0 < \theta < \pi/2).$$

We infer that

$$\text{(1)} \qquad \frac{\sin \theta}{\theta} \geq \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi} \qquad (0 < \theta \leq \pi/2).$$

Equality holds in (1) if and only if $\theta = \pi/2$.

On the other side we have

$$\sin \theta \leq \theta \qquad (\theta \geq 0).$$

Combining this inequality with (1), we get the double inequality

$$\text{(2)} \qquad \frac{2}{\pi} \leq \frac{\sin \theta}{\theta} < 1 \qquad (0 < |\theta| \leq \pi/2).$$

Inequality (1) is known as JORDAN's inequality.

Remark 1. Inequality (2) is an immediate consequence of the concavity of $\theta \mapsto \sin \theta$ on the interval $[0, \pi/2]$. The straight line $y = \frac{2}{\pi}\theta$ is a chord of $y = \sin \theta$, which joins the points $(0, \theta)$ and $(\pi/2, 1)$. The straight line $y = \theta$ is a tangent to $y = \sin \theta$ at the origin. Hence, the graph of $y = \sin \theta$ $(0 \leq \theta \leq \pi/2)$ lies between these straight lines.

Remark 2. Inequality

$$\text{(3)} \qquad \frac{\sin \theta}{\theta} \geq \frac{\pi^2 - \theta^2}{\pi^2 + \theta^2},$$

where θ is real, is due to R. REDHEFFER.

Inequalities (1) and (3) do not imply each other.

Reference

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2.4 Bernoulli's Inequality and its Generalizations

Theorem 1. *If $x > -1$ and if n is a positive integer, then*

$$(1) \qquad (1+x)^n \geq 1 + nx.$$

This inequality is called BERNOULLI's inequality.

Proof. If $n = 1$, then (1) is an equality. Let us assume that (1) holds if $n = k \geq 1$, i.e., that for $x > -1$,

$$(2) \qquad (1+x)^k \geq 1 + kx.$$

Multiplying (2) by $1 + x$ (> 0), we get

$$(1+x)^{k+1} \geq (1+x)(1+kx) = 1 + (k+1)x + kx^2,$$

whence

$$(3) \qquad (1+x)^{k+1} \geq 1 + (k+1)x.$$

This completes the induction proof.

For $-1 < x \neq 0$ the following generalizations of BERNOULLI's inequality are valid:

$$(4) \qquad (1+x)^a > 1 + ax \quad \text{if} \quad a > 1 \quad \text{or} \quad a < 0,$$

and

$$(5) \qquad (1+x)^a < 1 + ax \quad \text{if} \quad 0 < a < 1.$$

Indeed, by TAYLOR's formula

$$(6) \qquad (1+x)^a - 1 - ax = \frac{a(a-1)x^2}{2}(1+\theta x)^{a-2} \qquad (0 < \theta < 1).$$

By our assumptions, $1 + \theta x > 0$, and

$$\operatorname{sgn}\big((1+x)^a - 1 - ax\big) = \operatorname{sgn}\big(a(a-1)\big) \qquad (x \neq 0),$$

so that (4) and (5) are true.

Remark. Inequality (1) holds also if $-2 \leq x \leq -1$. Indeed, for such x we have

$$(1+x)^n \geq -|1+x|^n \geq -|1+x| = 1 + x \geq 1 + nx.$$

Theorem 2. *If $n = 2, 3, \ldots$ and $-1 < x < \frac{1}{n-1}$, then*

$$(7) \qquad (1+x)^n \leq 1 + \frac{nx}{1+(1-n)x},$$

with equality holding if and only if $x = 0$.

Proof. Applying (1) to $\left(1 - \frac{x}{x+1}\right)^n$, we obtain

$$(8) \qquad \left(1 - \frac{x}{1+x}\right)^n \geq 1 - n\frac{x}{1+x},$$

where $n = 1, 2, \ldots$, and $-\frac{x}{1+x} > -1$ since $x > -1$. However

$$1 - \frac{x}{1+x} = \frac{1}{1+x},$$

so that (8) is equivalent to

$$\frac{1}{(1+x)^n} \geq \frac{1+x-nx}{1+x}.$$

If $1 + x - nx > 0$ and $n = 2, 3, \ldots$ $\left(\text{i.e., } x < \frac{1}{n-1}\right)$, then

$$(1+x)^n \leq \frac{1+x}{1+x-nx} = 1 + \frac{nx}{1+x-nx}.$$

Therefore, inequality (7) is established.

The following generalization of BERNOULLI's inequality is easy to prove.

Theorem 3. *If each of real numbers x_i $(i = 1, \ldots, n)$ is greater than -1, and either all are positive or all negative, then*

$$(1+x_1)(1+x_2)\cdots(1+x_n) > 1 + x_1 + x_2 + \cdots + x_n.$$

N. HADŽIIVANOV and I. PRODANOV [1], among other things, proved:

Theorem 4. *Let $f_n(x) = (1+x)^n - 1 - nx$, where $n > 1$ is an odd integer. Then $f_n(x) = 0$ has only one root x_n, such that*

$$-3 \leq x_n < -2 - \frac{1}{n} \quad \textit{and} \quad x_n < x_{n+2} \quad (n = 3, 5, 7, \ldots),$$

and

$$f_n(x) > 0 \quad \textit{for} \quad x > x_n \quad (x \neq 0),$$

$$f_n(x) < 0 \quad \textit{for} \quad x < x_n.$$

If n is even, then $f_n(x) > 0$ holds for every real $x \neq 0$.

Another extension of inequality (1) is given by the following result.

Theorem 5. *Let*

$$F(k, a, x) = 1 + ax + C(a, 2)\, x^2 + \cdots + C(a, k)\, x^k$$

be the k-th partial sum of the binomial series for $(1+x)^a$, where $x > -1$. Then, if the first omitted term is

1° *positive, then $(1+x)^a > F(k, a, x)$,*

2° *zero, then $(1+x)^a = F(k, a, x)$,*

3° *negative, then $(1+x)^a < F(k, a, x)$.*

This result is due to L. GERBER [2], who at the end of his short note establishes the following: These inequalities can be used to generate inequalities for functions which are integrals of binomials such as the logarithm, the inverse trigonometric functions and elliptic integrals.

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2.5 Čebyšev's and Related Inequalities

Theorem 1. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are two real sequences such that*

(1) $a_1 \leq \cdots \leq a_n$ *and* $b_1 \leq \cdots \leq b_n$, *or* $a_1 \geq \cdots \geq a_n$ *and* $b_1 \geq \cdots \geq b_n$,

then the following inequality is true

$$\left(\frac{1}{n}\sum_{\nu=1}^{n} a_\nu\right)\left(\frac{1}{n}\sum_{\nu=1}^{n} b_\nu\right) \leq \frac{1}{n}\sum_{\nu=1}^{n} a_\nu b_\nu. \tag{2}$$

It is called the ČEBYŠEV inequality.

Proof. Using the simplified notations

$$\sum a = \sum_{k=1}^{n} a_k, \quad \sum b = \sum_{k=1}^{n} b_k, \quad \sum ab = \sum_{k=1}^{n} a_k b_k,$$

we have

$$\sum_{\mu}\sum_{\nu}(a_\mu b_\mu - a_\mu b_\nu) = \sum_{\mu}(n a_\mu b_\mu - a_\mu \sum b) = n\sum ab - \sum a \sum b,$$

$$\sum_{\mu}\sum_{\nu}(a_\nu b_\nu - a_\nu b_\mu) = \sum_{\nu}(n a_\nu b_\nu - a_\nu \sum b) = n\sum ab - \sum a \sum b.$$

Therefore

$$\begin{aligned} n\sum ab - \sum a \sum b &= \frac{1}{2}\sum_{\mu}\sum_{\nu}(a_\mu b_\mu - a_\mu b_\nu + a_\nu b_\nu - a_\nu b_\mu) \\ &= \frac{1}{2}\sum_{\mu}\sum_{\nu}(a_\mu - a_\nu)(b_\mu - b_\nu). \end{aligned} \tag{3}$$

Condition (1) implies

$$(a_\mu - a_\nu)(b_\mu - b_\nu) \geq 0 \text{ for } \mu, \nu = 1, \ldots, n.$$

Hence, from (3) we obtain

$$n\sum ab - \sum a \sum b \geq 0.$$

This inequality is equivalent to (2).

From (3) one can also deduce that equality holds in (2) if and only if $a_1 = \cdots = a_n$ or $b_1 = \cdots = b_n$.

Theorem 2. *If*

$$\begin{aligned} &0 \le a_1 \le \cdots \le a_n, \\ &0 \le b_1 \le \cdots \le b_n, \\ &\vdots \\ &0 \le l_1 \le \cdots \le l_n, \end{aligned} \tag{4}$$

then

$$\frac{\sum_{k=1}^{n} a_k}{n} \cdot \frac{\sum_{k=1}^{n} b_k}{n} \cdots \frac{\sum_{k=1}^{n} l_k}{n} \le \frac{\sum_{k=1}^{n} a_k b_k \cdots l_k}{n}. \tag{5}$$

As a special case, we have the following result: If $a_k \ge 0$ for $k = 1, \ldots, n$ and m is a positive integer, then

$$\left(\frac{1}{n} \sum_{k=1}^{n} a_k\right)^m \le \frac{1}{n} \sum_{k=1}^{n} a_k^m.$$

We mention that the restriction to nonnegative numbers in (4) is necessary if we consider three or more sequences. This fact is overlooked in a number of textbooks.

The conditions (1) that both sequences a and b are increasing, or both decreasing, give a sufficient condition for Čebyšev's inequality to hold. However, those conditions are not necessary. Some other sufficient conditions, which are also not necessary, were given by D. N. Labutin [1] though he did not mention Čebyšev's inequality explicitly.

This problem was completely solved by D. W. Sasser and M. L. Slater [2]. They gave the necessary and sufficient conditions for Čebyšev's inequality (2) to hold, by proving the following theorem.

Theorem 3. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sequences of real numbers. Let a and b be n-dimensional column vectors whose components are $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. Furthermore, let e be an n-dimensional column vector with all entries equal to* 1. *Then, a necessary and sufficient condition for* Čebyšev's *inequality to hold is $b = Aa + ce$ or $a = Ab + ce$, where c is a real number and A is a real positive semidefinite matrix, such that the sum of the elements of any column or row is* 0. *Equality occurs in* (2) *if and only if $(A + A')\,a = 0$ or $(A + A')\,b = 0$.*

Remark 1. A quadratic real matrix A is called positive semidefinite if and only if the quadratic form (x, Ax) is nonnegative for all real x.

The following inequality, proved by G. Seitz [3], contains both Cauchy's and Čebyšev's inequalities.

Theorem 4. *Let* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$ *and* $u = (u_1, \ldots, u_n)$ *be given sequences of real numbers, and let* a_{ij} $(i, j = 1, \ldots, n)$ *be given real numbers. If for every pair of numbers* i, j $(i < j)$ *and for every pair* r, s $(r < s)$

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \cdot \begin{vmatrix} z_r & z_s \\ u_r & u_s \end{vmatrix} \geq 0, \quad \text{and} \quad \begin{vmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{vmatrix} \geq 0,$$

then

$$\frac{\sum_{i,j=1}^{n} a_{ij} x_i z_j}{\sum_{i,j=1}^{n} a_{ij} x_i u_j} \geq \frac{\sum_{i,j=1}^{n} a_{ij} y_i z_j}{\sum_{i,j=1}^{n} a_{ij} y_i u_j}.$$

The following result of T. Popoviciu [4] is closely connected to Čebyšev's inequality.

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be nondecreasing sequences of real numbers and let x_{ij} $(i, j = 1, \ldots, n)$ be real numbers. Then necessary and sufficient conditions for the numbers x_{ij}, so that the inequality

$$F(a, b) = \sum_{i,j=1}^{n} x_{ij} a_i b_j \geq 0 \tag{6}$$

holds: 1° for all nondecreasing sequences a and b, or 2° for all nonnegative nondecreasing sequences a and b, are contained in the following two theorems.

Theorem 5. *With the condition* 1°, $F(a, b) \geq 0$ *if and only if*

$$\sum_{i=r}^{n} \sum_{j=s}^{n} x_{ij} \geq 0 \qquad (r = 1, \ldots, n;\ s = 2, \ldots, n),$$

$$\sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij} = 0 \qquad (r = 1, \ldots, n).$$

Theorem 6. *With the condition* 2°, $F(a, b) \geq 0$ *if and only if*

$$\sum_{i=r}^{n} \sum_{j=s}^{n} x_{ij} \geq 0 \qquad (r = 1, \ldots, n; s = 1, \ldots, n).$$

Remark 2. Theorems 5 and 6 refer only to the cases of nondecreasing sequences (Theorem 5) and nonnegative nondecreasing sequences (Theorem 6). Similar results are obtained in other cases.

Inequality (6) is a generalization of a number of known inequalities. For example, taking

$$x_{ij} = n - 1 \quad (i = j), \qquad x_{ij} = -1 \quad (i \neq j),$$

we get the inequality of Čebyšev.

H. W. McLaughlin and F. T. Metcalf [5] proved the following interesting result:

Theorem 7. *Let I and J denote nonempty disjoint finite sets of distinct positive integers. Suppose that (a_k) and (b_k), with $k \in I \cup J$, are sequences of nonnegative real numbers, (p_k), with $k \in I \cup J$, is a sequence of positive numbers, and $r > 0$. Define M_r and T_r by*

$$M_r(a; p, I) = \left(\frac{\sum\limits_{k \in I} p_k a_k^r}{\sum\limits_{k \in I} p_k}\right)^{\frac{1}{r}},$$

and

$$T_r(a, b; I) = \Big(\sum_{k \in I} p_k\Big)\big(M_r(ab; p, I)^r - M_r(a; p, I)^r\, M_r(b; p, I)^r\big).$$

If the pairs

$$(7) \qquad \big(M_r(a; p, I), M_r(a; p, J)\big) \quad \text{and} \quad \big(M_r(b; p, I), M_r(b; p, J)\big)$$

are similarly ordered, then

$$(8) \qquad T_r(a, b; I \cup J) \geq T_r(a, b; I) + T_r(a, b; J).$$

If the pairs (7) *are oppositely ordered, then the sense of* (8) *reverses. In both cases equality holds if and only if either*

$$M_r(a; p, I) = M_r(a; p, J), \quad \text{or} \quad M_r(b; p, I) = M_r(b; p, J).$$

An integral analogue of Čebyšev's inequality (2) is given by the following theorem:

Theorem 8. *Let f and g be real and integrable functions on $[a, b]$ and let them both be either increasing or decreasing. Then*

$$(9) \qquad \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx \geq \frac{1}{b-a}\int_a^b f(x)\, dx \cdot \frac{1}{b-a}\int_a^b g(x)\, dx.$$

If one function is increasing and the other decreasing, the reverse inequality holds.

The Čebyšev inequality (9) can be, as shown by O. Dunkel [6], generalized to n functions:

Theorem 9. *If $f_1, \ldots, f_n$ are nonnegative functions which are monotone in the same sense, and integrable on (a, b), then*

$$\int_a^b f_1(x)\, dx \cdots \int_a^b f_n(x)\, dx \leq (b-a)^{n-1} \int_a^b f_1(x) \cdots f_n(x)\, dx.$$

The following generalization of inequality (9) is due to P. L. Čebyšev [7]:

Theorem 10. *Let f and g be two functions which are integrable and monotone in the same sense on (a, b) and let p be a positive and integrable function on the same interval. Then*

$$\int_a^b p(x)\, f(x)\, g(x)\, dx \int_a^b p(x)\, dx \geq \int_a^b p(x)\, f(x)\, dx \int_a^b p(x)\, g(x)\, dx, \tag{10}$$

with equality if and only if one of the functions f, g reduces to a constant.

If f and g are monotone in the opposite sense, inequality in (10) *reverses.*

From (10) for $p(x) \equiv 1$ follows (9). For $p(x) = f_2(x)^2$, $f(x) = g(x) = f_1(x)/f_2(x)$, (10) yields the BUNIAKOWSKI-SCHWARZ inequality (see 2.6.1).

M. BIERNACKI [8] proved inequality (10) under different conditions from those given in Theorem 10. In fact, he proved:

Theorem 11. *Let f, g and p be integrable functions on (a, b), and let p be positive on that interval. If the functions*

$$f_1(x) = \frac{\int_a^x p(x) f(x)\, dx}{\int_a^x p(x)\, dx}, \qquad g_1(x) = \frac{\int_a^x p(x)\, g(x)\, dx}{\int_a^x p(x)\, dx}$$

reach extreme values only in a finite number of common points from (a, b), and if they are monotone in the same sense on (a, b), then (10) *holds.*

In the statement of Theorem 11 functions f_1 and g_1 can be replaced by the following functions:

$$f_2(x) = \frac{\int_x^b p(x)\, f(x)\, dx}{\int_x^b p(x)\, dx} \quad \text{and} \quad g_2(x) = \frac{\int_x^b p(x)\, g(x)\, dx}{\int_x^b p(x)\, dx}.$$

If the functions f_1 and g_1 (or f_2 and g_2) are monotone in the opposite sense, inequality in (10) is reversed.

Theorem 10 was rediscovered by M. FUJIWARA [9], in a somewhat different form. In fact, FUJIWARA's inequality follows from (10) setting $p(x) = f_2(x)\, g_2(x)$, $f(x) = f_1(x)/f_2(x)$ and $g(x) = g_1(x)/g_2(x)$. In connection with this, see also certain results of S. ISAYAMA [10] and T. HAYASHI [11].

The paper [12] of N. A. SAPOGOV is closely related to Theorem 10. Besides, the same paper contains a very simple proof of ČEBYŠEV's inequality and also a number of references concerning the topics in question.

Some generalizations of (10) are also due to J. SHOHAT [13].

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2.6 Cauchy's and Related Inequalities

2.6.1 Some Refinements and Extensions of Cauchy's Inequality

First of all we shall give a new proof of the following important Theorems 1 and 2, which were already proved in 2.1.2.

Theorem 1. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{1}$$

Equality holds if and only if the sequences a and b are linearly dependent.

Proof. Starting with the LAGRANGE identity,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2,$$

which holds for real numbers, we obtain (1).

Since CAUCHY's inequality (1) (see [1]) follows from the above LAGRANGE identity, it is also called LAGRANGE's inequality.

Theorem 2. *For arbitrary complex sequences* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$, *we have*

$$\left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right) \geq \left|\sum_{i=1}^{n} a_i b_i\right|^2, \tag{2}$$

with equality holding if and only if the sequences $\bar{a}$ *and* b *are proportional.*

Proof. By application of the BINET-CAUCHY identity for determinants one can obtain the following identity

$$\left(\sum_{i=1}^{n} a_i c_i\right)\left(\sum_{i=1}^{n} b_i d_i\right) - \left(\sum_{i=1}^{n} a_i d_i\right)\left(\sum_{i=1}^{n} b_i c_i\right) = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i).$$

Replacing a_i, b_i, c_i, d_i by $\bar{a}_i, b_i, a_i, \bar{b}_i$ respectively, we get

$$\left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right) - \left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{1 \leq i < j \leq n} |\bar{a}_i b_j - a_j \bar{b}_i|^2.$$

From this identity follows (2).

For a stronger inequality than (2), see 3.8.5.

In the following we give some inequalities which generalize or refine CAUCHY's inequality.

Theorem 3. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are sequences of real numbers and* $0 \leq x \leq 1$, *then*

$$\left(\sum_{k=1}^{n} a_k b_k + x \sum_{i \neq j} a_i b_j\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 + 2x \sum_{i<j} a_i a_j\right)\left(\sum_{k=1}^{n} b_k^2 + 2x \sum_{i<j} b_i b_j\right).$$

For $x = 0$, this inequality reduces to CAUCHY's inequality.

A simple proof of this inequality, due to S. S. WAGNER, was given by P. FLOR [2].

D. K. CALLEBAUT [3] proved:

Theorem 4. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are sequences of positive numbers and* $1 \leq z \leq y \leq 2$ *or* $0 \leq y \leq z \leq 1$, *then*

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \geq \left(\sum_{k=1}^{n} a_k^y b_k^{2-y}\right)\left(\sum_{k=1}^{n} a_k^{2-y} b_k^y\right)$$
$$\geq \left(\sum_{k=1}^{n} a_k^z b_k^{2-z}\right)\left(\sum_{k=1}^{n} a_k^{2-z} b_k^z\right) \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

A simple proof of these inequalities, which interpolate CAUCHY's inequality, was given by H. W. MCLAUGHLIN and F. T. METCALF [4]. In fact, they showed that the above inequality is a consequence of HÖLDER's inequality (see 2.8).

In [5] C. J. ELIEZER and D. E. DAYKIN present a general extension of CALLEBAUT's result.

In the case when the considered sequences a and b have an even number of terms, H. W. MCLAUGHLIN [6], p. 66, has proved the following inequality which refines CAUCHY's inequality.

Theorem 5. *If $a = (a_1, \ldots, a_{2n})$ and $b = (b_1, \ldots, b_{2n})$ are sequences of real numbers, then*

$$\left(\sum_{k=1}^{2n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{2n} a_k^2\right)\left(\sum_{k=1}^{2n} b_k^2\right) - \left(\sum_{k=1}^{n} a_{2k} b_{2k-1} - a_{2k-1} b_{2k}\right)^2.$$

H. W. MCLAUGHLIN [6], p. 70, has also proved an analogous inequality for the case when the sequences a and b have $4n$ terms each.

Notice that an inequality analogous to inequality (1) holds for quaternions. This result was obtained by H. W. MCLAUGHLIN [6], p. 24, and it reads:

Theorem 6. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be quaternions. Then*

$$\left\|\sum_{k=1}^{n} a_k \bar{b}_k\right\|^2 \leq \left(\sum_{k=1}^{n} \|a_k\|^2\right)\left(\sum_{k=1}^{n} \|b_k\|^2\right)$$

with equality if and only if $a_1 = \cdots = a_n = 0$, or $b_k = \lambda a_k$ for real λ and for $k = 1, \ldots, n$.

Remark. If $a = R(a) + iI(a) + jJ(a) + kK(a)$, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, then the conjugate $\bar{a}$ of the quaternion a is given by

$$\bar{a} = R(a) - iI(a) - jJ(a) - kK(a),$$

and the norm $\|a\|$ of a is defined by

$$\|a\|^2 = R(a)^2 + I(a)^2 + J(a)^2 + K(a)^2.$$

For other generalizations of CAUCHY's inequality see 2.6.2 and 2.8.

We shall now prove an integral analogue of CAUCHY's inequality:

Theorem 7. *Let f and g be real and integrable functions on $[a, b]$. Then*

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leq \left(\int_a^b f(x)^2\, dx\right)\left(\int_a^b g(x)^2\, dx\right), \tag{3}$$

with equality holding if and only if f and g are linearly dependent functions.

Proof. For every real number t,

$$\int_a^b (tf(x) + g(x))^2\, dx \geq 0,$$

i.e.,

$$t^2 \int_a^b f(x)^2\,dx + 2t \int_a^b f(x)\,g(x)\,dx + \int_a^b g(x)^2\,dx \geq 0,$$

which directly implies (3).

Inequality (3) is called the SCHWARZ or, more correctly, the BUNIAKOWSKI-SCHWARZ inequality.

Let us finally mention a result of A. SIGNORINI [7], which is a consequence of the BUNIAKOWSKI-SCHWARZ inequality.

Theorem 8. *Let f_1, f_2, p_1, p_2 be almost everywhere continuous functions in the region V, and let $p_2 \geq p_1 > 0$ but not $p_1 \equiv p_2$. Then*

$$\int_V p_2 f_1^2\,dV \int_V p_2 f_2^2\,dV - \int_V p_2 f_1 f_2\,dV$$
$$\geq \int_V p_1 f_1^2\,dV \int_V p_1 f_2^2\,dV - \int_V p_1 f_1 f_2\,dV \geq 0.$$

There are a number of papers which extend the CAUCHY inequality in the BANACH, HILBERT, or other spaces. See, particularly, the papers [8], [9], [10] of S. KUREPA, [11] of J. B. DIAZ and F. T. METCALF, [12] of E. R. LORCH, and [13] of F. D. MURNAGHAN.

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2.6.2 Gram's Inequality

Let $x_1, \ldots, x_n$ be vectors of a unitary space X. Then

$$G(x_1, \ldots, x_n) = \begin{Vmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \\ (x_n, x_1) & & (x_n, x_n) \end{Vmatrix}$$

is called the GRAM matrix of vectors $x_1, \ldots, x_n$.

The determinant $\Gamma(x_1, \ldots, x_n) = \det G(x_1, \ldots, x_n)$ is called the GRAM determinant of vectors $x_1, \ldots, x_n$.

Theorem 1. *The following inequality*

$$\Gamma(x_1, \ldots, x_n) \geq 0, \tag{1}$$

holds with equality if and only if the vectors $x_1, \ldots, x_n$ are linearly dependent.

This inequality is called GRAM's inequality.

Proof. If the vectors $x_1, \ldots, x_n$ are linearly dependent, then there exist scalars $\alpha_1, \ldots, \alpha_n$ not all zero, such that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0. \tag{2}$$

Multiplying the above equation by x_k ($k = 1, \ldots, n$), we get

$$\alpha_1 (x_1, x_k) + \cdots + \alpha_n (x_n, x_k) = 0 \qquad (k = 1, \ldots, n). \tag{3}$$

System (3) will have nontrivial solutions for α_i if $\Gamma(x_1, \ldots, x_n) = 0$. Therefore, if the vectors $x_1, \ldots, x_n$ are linearly dependent, then $\Gamma(x_1, \ldots, x_n) = 0$. Conversely, if $\Gamma(x_1, \ldots, x_n) = 0$, then the system (3) has nontrivial solutions for α_i. Let us prove that the equality (2) holds for those values of α_i.

System (3) can be written in the form

$$(\alpha_1 x_1 + \cdots + \alpha_n x_n, x_k) = 0 \qquad (k = 1, \ldots, n).$$

Multiplying by $\overline{\alpha_k}$ and adding, we get

$$\sum_{k=1}^{n} \overline{\alpha_k} (\alpha_1 x_1 + \cdots + \alpha_n x_n, x_k) = 0,$$

i.e.,

$$(\alpha_1 x_1 + \cdots + \alpha_n x_n, \ \alpha_1 x_1 + \cdots + \alpha_n x_n) = 0.$$

This implies the equality (2).

We have, therefore, proved that equality holds in (1) if and only if the vectors $x_1, \ldots, x_n$ are linearly dependent.

Let us now prove that $\Gamma(x_1, \ldots, x_n) > 0$ if the vectors $x_1, \ldots, x_n$ are linearly independent. Let

$$y_r = \left|\begin{array}{c:c} G(x_1, \ldots, x_{r-1}) & \begin{matrix} x_1 \\ \vdots \\ x_{r-1} \end{matrix} \\ \hdashline (x_r, x_1) \cdots (x_r, x_{r-1}) & x_r \end{array}\right|. \tag{4}$$

Then

$$(y_r, x_s) = \left|\begin{array}{c:c} G(x_1, \ldots, x_{r-1}) & \begin{matrix} (x_1, x_s) \\ \vdots \\ (x_{r-1}, x_s) \end{matrix} \\ \hdashline (x_r, x_1) \cdots (x_r, x_{r-1}) & (x_r, x_s) \end{array}\right|. \tag{5}$$

If $s < r$, then there are two identical columns in (4), and therefore $(y_r, x_s) = 0$.

Multiplying (4) by y_r and taking into account that $(y_r, x_s) = 0$ $(s < r)$, we get

$$|y_r|^2 = \Gamma(x_1, \ldots, x_{r-1}) \overline{\Gamma(x_1, \ldots, x_r)}.$$

Specifically, for $r = 2$,

$$|y_2|^2 = \Gamma(x_1) \overline{\Gamma(x_1, x_2)},$$

i.e.,

$$\overline{\Gamma(x_1, x_2)} = \frac{|y_2|^2}{|x_1|^2}.$$

Therefore $\overline{\Gamma(x_1, x_2)} > 0$, and so $\Gamma(x_1, x_2) > 0$. It can then easily be shown by induction that

$$\Gamma(x_1, \ldots, x_n) > 0.$$

In many books, in some or more detail, the GRAM inequality is exposed. See, for instance, [1], pp. 345–357, or [2], pp. 176–187.

We give some special inequalities which involve GRAM's determinant (see [1], pp. 382–383).

If $x_1, \ldots, x_n$ are vectors of a unitary space X, then, for $1 \le k < n$,

$$\Gamma(x_1, \ldots, x_n) \le |x_1|^2 \cdots |x_n|^2;$$

$$\frac{\Gamma(x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_k)} \le \frac{\Gamma(x_2, \ldots, x_n)}{\Gamma(x_2, \ldots, x_k)} \le \cdots \le \Gamma(x_{k+1}, \ldots, x_n);$$

$$\Gamma(x_1, \ldots, x_n) \leq \Gamma(x_1, \ldots, x_k)\, \Gamma(x_{k+1}, \ldots, x_n);$$

$$\left|\begin{array}{c:c} G(x_1, \ldots, x_n) & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hdashline 1 \cdots 1 & 0 \end{array}\right| \leq 0.$$

The following inequality also holds for GRAM's determinant (see [2], pp. 184—185):

$$\Gamma(x_1 + y_1, x_2, \ldots, x_n)^{1/2} \leq \Gamma(x_1, x_2, \ldots, x_n)^{1/2} + \Gamma(y_1, x_2, \ldots, x_n)^{1/2}.$$

If the vectors $x_1, \ldots, x_n$ are elements of an arbitrary HILBERT space, then we have (see [3] and [4]):

Theorem 2. *Given the vectors $x_1, \ldots, x_n$ and a projection P with $y_i = Px_i$ $(i = 1, \ldots, n)$, then*

$$\Gamma(x_1, \ldots, x_n) \geq \Gamma(y_1, \ldots, y_n),$$

with equality if and only if $x_i = y_i$ $(i = 1, \ldots, n)$.

Theorem 3. *Given the vectors $x_1, \ldots, x_n$ and a projection P with $y_i = Px_i$ $(i = 1, \ldots, n)$, then*

$$\frac{\Gamma(y_1, \ldots, y_{n-1})}{\Gamma(y_1, \ldots, y_n)} \geq \frac{\Gamma(x_1, \ldots, x_{n-1})}{\Gamma(x_1, \ldots, x_n)},$$

where $\Gamma(y_1, \ldots, y_n) \cdot \Gamma(x_1, \ldots, x_n) \neq 0$.

Concerning other inequalities involving the GRAM determinant, consult also the interesting paper [5] of F. T. METCALF. This paper contains at the end a list of literature of 19 items.

An integral interpretation of (1) is given by

Theorem 4. *Let $f_1, \ldots, f_n$ be real and integrable functions on $[a, b]$. Then*

$$\begin{vmatrix} \int_a^b f_1(x)^2\, dx & \int_a^b f_1(x)\, f_2(x)\, dx & \cdots & \int_a^b f_1(x)\, f_n(x)\, dx \\ \int_a^b f_2(x)\, f_1(x)\, dx & \int_a^b f_2(x)^2\, dx & & \int_a^b f_2(x)\, f_n(x)\, dx \\ \vdots & & & \\ \int_a^b f_n(x)\, f_1(x)\, dx & \int_a^b f_n(x)\, f_2(x)\, dx & & \int_a^b f_n(x)^2\, dx \end{vmatrix} \geq 0.$$

For this inequality see, for example, the paper [6] of L. TOCCHI.

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2.7 Young's Inequality

Theorem 1. *Let f be a real-valued, continuous and strictly increasing function on $[0, c]$, with $c > 0$. If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then*

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab, \tag{1}$$

where f^{-1} is the inverse function of f.

Equality holds in (1) *if and only if $b = f(a)$.*

Since this result is due to W. H. Young [1], (1) is called Young's inequality.

Proof. We set

$$g(a) = ab - \int_0^a f(x)\,dx, \tag{2}$$

and consider $b > 0$ as a parameter. Since $g'(a) = b - f(a)$, and f is strictly increasing, we have

$$\begin{aligned} g'(a) &> 0 \quad \text{for} \quad 0 < a < f^{-1}(b), \\ g'(a) &= 0 \quad \text{for} \quad a = f^{-1}(b), \\ g'(a) &< 0 \quad \text{for} \quad a > f^{-1}(b). \end{aligned}$$

Hence, $g(a)$ is a maximum of g for $a = f^{-1}(b)$. Therefore, we have

$$g(a) \leq \max g(x) = g\big(f^{-1}(b)\big). \tag{3}$$

Integrating by parts, we obtain

$$g\big(f^{-1}(b)\big) = bf^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,dx = \int_0^{f^{-1}(b)} xf'(x)\,dx.$$

Substituting $y = f(x)$, the above integral becomes

$$g\big(f^{-1}(b)\big) = \int_0^b f^{-1}(y)\,dy. \tag{4}$$

Putting expressions (2) and (4) into (3), we get (1).

Geometric interpretation. The area of the curvilinear triangle OAP is given by $\int_0^a f(x)\,dx$, and the area of the curvilinear triangle ORB by $\int_0^b f^{-1}(x)\,dx$.

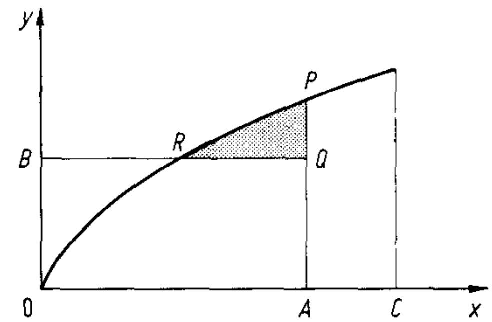

Fig. 3

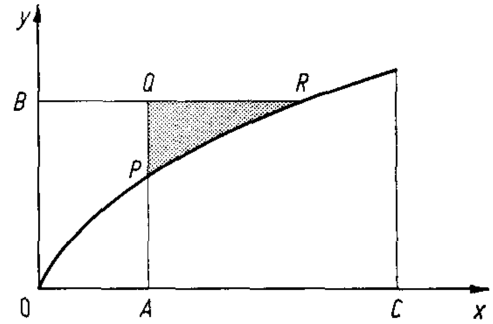

Fig. 4

Figs. 3 and 4 affirm that inequality (1) is justified.

Example 1. The function $x \mapsto f(x) = x^{p-1}$ $(p > 1)$ in each interval $(0, c)$ $(c > 0)$ satisfies all the conditions assumed in Theorem 1. Applying (1), we get

$$\int_0^a x^{p-1}\,dx + \int_0^b x^{\frac{1}{p-1}}\,dx \geq ab, \quad \text{i.e.,} \quad \frac{1}{p}\,a^p + \frac{p-1}{p}\,b^{\frac{p}{p-1}} \geq ab.$$

The latter inequality is usually written in the form

$$(5) \qquad \frac{1}{p}\,a^p + \frac{1}{q}\,b^q \geq ab, \quad \text{where} \quad a, b \geq 0;\, p > 1;\ \frac{1}{p} + \frac{1}{q} = 1.$$

Remark 1. Inequality (5) is, in fact, the inequality between the arithmetic and geometric weighted means (see 2.1.1 and 2.14.2). For another proof of this fundamental inequality consult paper [2] of F. RIESZ.

Example 2. The function $x \mapsto \log(1 + x)$ also satisfies all the conditions imposed by Theorem 1. Applying (1) we get

$$\int_0^a \log\,(1 + x)\,dx + \int_0^b (e^x - 1)\,dx \geq ab,$$

i.e.,

$$(1 + a)\log\,(1 + a) - (1 + a) + (e^b - b) \geq ab \quad \text{with} \quad a, b \geq 0.$$

Theorem 2. *Let f and g be positive functions, with f' and g' nonnegative and continuous on $[0, b]$. Let $f(0) = 0$. Then, for $0 < a \leq b$,*

$$f(a)\,g(b) \leq \int_0^a g(x)\,f'(x)\,dx + \int_0^b f(x)\,g'(x)\,dx,$$

with equality if and only if $a = b$, or $a < b$ but g constant on (a, b).

Hint for proof. Integrate by parts the first integral on the right-hand side.

Remark 2. The above inequality is similar to YOUNG's inequality, but more elementary.

Remark 3. The above result was communicated to us by P. R. BEESACK.

Among others, R. COOPER [3] and A. OPPENHEIM [4] gave some extensions of YOUNG's inequality, from which we quote that of OPPENHEIM.

Theorem 3. *Let $f_1, \ldots, f_n$ be continuous, nonnegative and strictly increasing functions, at least one of which is zero at $x = 0$. Then*

$$\prod_{k=1}^{n} f_k(t_k) \leq \sum_{k=1}^{n} \int_0^{t_k} \prod_{\substack{p,q=1\\ p\neq q}}^{n} f_p(x)\, df_q(x),$$

where $t_k \geq 0$ for $k = 1, \ldots, n$, and where the integrals involved are STIELTJES'.

T. TAKAHASHI [5] has proved the converse of YOUNG's inequality. His result reads:

Theorem 4. *If for $x \geq 0$ f and g are continuous and increasing functions such that $f(0) = g(0) = 0$, $g^{-1}(x) \geq f(x)$ for all $x \geq 0$, and if for every $a > 0$ and $b > 0$ we have*

$$ab \leq \int_0^a f(x)\, dx + \int_0^b g(x)\, dx,$$

then f and g are inverse.

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2.8 Hölder's Inequality

Theorem 1. *If $a_k \geq 0$, $b_k \geq 0$ for $k = 1, \ldots, n$, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \geq \sum_{k=1}^{n} a_k b_k, \tag{1}$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ *for* $k = 1, \ldots, n$, *where* α *and* β *are real nonnegative constants with* $\alpha^2 + \beta^2 > 0$.

This inequality is called HÖLDER's inequality (see [1]).

Proof. If $\sum_{k=1}^{n} a_k^p = 0$ or $\sum_{k=1}^{n} b_k^q = 0$, then equality holds in (1). Assume now that $\sum_{k=1}^{n} a_k^p > 0$ and $\sum_{k=1}^{n} b_k^q > 0$. Putting

$$a = a_\nu \left(\sum_{k=1}^{n} a_k^p\right)^{-1/p}, \qquad b = b_\nu \left(\sum_{k=1}^{n} b_k^q\right)^{-1/q} \tag{2}$$

into inequality

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab \quad \left(\frac{1}{p} + \frac{1}{q} = 1,\ p > 1 \ \text{ and } \ a, b \geq 0\right), \tag{3}$$

we get

$$\frac{1}{p} \frac{a_\nu^p}{\sum_{k=1}^{n} a_k^p} + \frac{1}{q} \frac{b_\nu^p}{\sum_{k=1}^{n} b_k^q} \geq \frac{a_\nu b_\nu}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}.$$

Adding together these inequalities for $\nu = 1, \ldots, n$, we have

$$\frac{1}{p} + \frac{1}{q} \geq \frac{\sum_{k=1}^{n} a_k b_k}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}.$$

This inequality is equivalent to (1) since $\frac{1}{p} + \frac{1}{q} = 1$.

Since equality holds in (3) if and only if $a^p = b^q$, we conclude, in virtue of (2), that there is equality in (1) if and only if $\left(\sum_{k=1}^{n} a_k^p\right)^{-1} \cdot a_k^p = \left(\sum_{k=1}^{n} b_k^q\right)^{-1} \cdot b_k^q$ for $k = 1, \ldots, n$, i.e., if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1, \ldots, n$.

Theorem 2. *If* $a_k > 0$ *and* $b_k > 0$ *for* $k = 1, \ldots, n$, *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p < 0$ *or* $q < 0$, *then*

$$\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}} \leq \sum_{k=1}^{n} a_k b_k, \tag{4}$$

with equality if and only if $\alpha a_k^p = \beta b_k^q$ *for* $k = 1, \ldots, n$, *where* α *and* β *are real nonnegative constants with* $\alpha^2 + \beta^2 > 0$.

Proof. Assume that $p < 0$, and put $P = -p/q$, $Q = 1/q$. Then $1/P + 1/Q = 1$ with $P > 0$ and $Q > 0$. Therefore, according to (1), we have

$$\left(\sum_{k=1}^{n} A_k^P\right)^{\frac{1}{P}} \left(\sum_{k=1}^{n} B_k^Q\right)^{\frac{1}{Q}} \geq \sum_{k=1}^{n} A_k B_k,$$

where $A_k > 0$ and $B_k > 0$ for $k = 1, \ldots, n$.

The last inequality for $A_k = a_k^{-q}$ and $B_k = a_k^q b_k^q$ becomes (4).

J. L. W. V. Jensen [2] proved the following generalization of Hölder's inequality:

Theorem 3. *Let a_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ be positive numbers and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_m} \geq 1$. Then*

$$\sum_{i=1}^{n} a_{i1} \cdots a_{im} \leq \left(\sum_{i=1}^{n} a_{i1}^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \cdots \left(\sum_{i=1}^{n} a_{im}^{\alpha_m}\right)^{\frac{1}{\alpha_m}}.$$

The proof of the above theorem will be given in 2.14.2.

E. F. Beckenbach [3] gave the following generalization of Hölder's inequality:

Theorem 4. *Let $c = (c_1, \ldots, c_m)$ and $k = (k_1, \ldots, k_n)$ be positive vectors such that $0 < m < n$, and p and q be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for positive numbers $x_{m+1}, x_{m+2}, \ldots, x_n$ and for $p > 1$,*

$$\frac{\left(\sum_{i=1}^{m} c_i^p + \sum_{i=m+1}^{n} x_i^p\right)^{\frac{1}{p}}}{\sum_{i=1}^{m} c_i k_i + \sum_{i=m+1}^{n} x_i k_i} \geq \frac{\left(\sum_{i=1}^{m} c_i^p + \sum_{i=m+1}^{n} \bar{c}_i^p\right)^{\frac{1}{p}}}{\sum_{i=1}^{m} c_i k_i + \sum_{i=m+1}^{n} \bar{c}_i k_i}, \tag{5}$$

where

$$\bar{c}_i = \left(\frac{\sum_{j=1}^{m} c_j^p}{\sum_{j=1}^{m} c_j k_j} k_i\right)^{\frac{q}{p}} \qquad (i = m + 1, \ldots, n).$$

Equality holds in (5) *if and only if*

$$x_i = \bar{c}_i \qquad (i = m + 1, \ldots, n). \tag{6}$$

Inequality is reversed in (5) *if $p < 1$ and $p \neq 0$, with equality holding if and only if condition* (6) *is fulfilled.*

For $m = 1$, from this theorem we get Hölder's inequality.

For some other results analogous to the above, consult also the interesting paper [4] of E. F. Beckenbach.

In all the above theorems we have assumed that $\frac{1}{p}+\frac{1}{q}=1$. In the case where $\frac{1}{p}+\frac{1}{q}<1$, D. E. DAYKIN and C. J. ELIEZER [5] proved some inequalities analogous to HÖLDER's inequality which they have formulated in

Theorem 5. 1° *If* $a_k > 0$ *and* $b_k > 0$ *for* $k = 1, \ldots, n$ *and* $\frac{1}{p}+\frac{1}{q}<1$, *then*

$$2\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} a_k^{2-p}\, b_k^2\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} a_k^2\, b_k^{2-q}\right)^{\frac{1}{q}}.$$

2° *If* $a_k < 1$, $b_k < 1$ *for* $k = 1, \ldots, n$, *or if* $P = \prod_{i,j=1}^{n} (a_i a_j b_i b_j)^{a_i a_j b_i b_j} < 1$ *and* $\frac{1}{p}+\frac{1}{q}<1$, *then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \left(\sum_{k=1}^{n} a_k^{2-p} b_k^2\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} a_k^2 b_k^{2-q}\right)^{\frac{1}{q}}.$$

3° *If* $a_k > 1$, $b_k > 1$ *for* $k = 1, \ldots, n$, *or if* $P > 1$ (P is defined as in 2°) *and* $\frac{1}{p}+\frac{1}{q}<1$, *then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}.$$

For complex numbers we have:

Theorem 6. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are complex vectors and* $1 < p < +\infty$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}, \tag{7}$$

with equality if and only if the sequences $(|a_k|^p)$ *and* $(|b_k|^q)$ *for* $k = 1, \ldots, n$ *are proportional, and* $\arg a_k b_k$ *is independent of* k.

The conditions of equality in (7) can be obtained using, for instance, the three lines theorem (see 1.4.5). This can be found, for example, in Thesis of G. O. THORIN [6].

Let us now quote an integral analogue of HÖLDER's inequality (1).

Theorem 7. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$, then*

$$\int_a^b |f(x)\, g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A\,|f(x)|^p = B\,|g(x)|^q$ almost everywhere, where A and B are constants.

Notice that W. N. Everitt [7] obtained some interesting results related to the integral form of Hölder's inequality. We shall quote one of these results.

Theorem 8. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let E and E' with $E' \subset E$ be* Lebesgue*-measurable linear sets. If f_1 and f_2 are complex measurable functions such that $f_1 \in L^p(E)$, $f_2 \in L^q(E)$, define the function H by*

$$H(E) = \left(\int_E |f_1(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_E |f_2(x)|^q\, dx\right)^{\frac{1}{q}} - \left|\int_E f_1(x)\, f_2(x)\, dx\right|.$$

Then

$$0 \leq H(E - E') \leq H(E) - H(E').$$

Finally, we note the following:

From (1), for $p = \frac{r-t}{r-s}$, $q = \frac{r-t}{s-t}$ $(r > s > t > 0)$, $a_k^p = p_k x_k^t$, $b_k^q = p_k x_k^r$ ($p_k \geq 0$, $x_k \geq 0$ for $k = 1, \ldots, n$), follows

$$\left(\sum_{k=1}^n p_k x_k^s\right)^{r-t} \leq \left(\sum_{k=1}^n p_k x_k^t\right)^{r-s} \left(\sum_{k=1}^n p_k x_k^r\right)^{s-t}.$$

This is called Lyapunov's inequality.

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2.9 Minkowski's and Related Inequalities

Theorem 1. *If* $a_k \geq 0$ *and* $b_k \geq 0$, *for* $k = 1, \ldots, n$, *and* $p > 1$, *then*

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}, \tag{1}$$

with equality holding if and only if the sequences $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are proportional.*

This inequality carries the name of H. Minkowski.

Proof. We start with the identity

$$(a_k + b_k)^p = a_k(a_k + b_k)^{p-1} + b_k(a_k + b_k)^{p-1}.$$

Summing over $k = 1, \ldots, n$, we get

$$\sum_{k=1}^{n}(a_k + b_k)^p = \sum_{k=1}^{n} a_k(a_k + b_k)^{p-1} + \sum_{k=1}^{n} b_k(a_k + b_k)^{p-1}. \tag{2}$$

By Hölder's inequality, for $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, we have

$$\sum_{k=1}^{n} a_k(a_k + b_k)^{p-1} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n}(a_k + b_k)^{q(p-1)}\right)^{1/q},$$

$$\sum_{k=1}^{n} b_k(a_k + b_k)^{p-1} \leq \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\left(\sum_{k=1}^{n}(a_k + b_k)^{q(p-1)}\right)^{1/q}.$$

Using $q(p-1) = p$ and adding the last two relations, we obtain

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right)\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/q}.$$

Dividing both sides of the above inequality by $\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/q}$, we get (1).

D. E. Daykin and C. J. Eliezer [1], as a consequence of a more general result, proved a number of inequalities contained in

Theorem 2. *Let* $a_k \geq 0$, $b_k \geq 0$ *for* $k = 1, \ldots, n$. *If* $p > 1$, *then*

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \geq \left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p}, \tag{3}$$

and

$$\left(\sum_{k=1}^{n} a_k^{1/p}\right)^p + \left(\sum_{k=1}^{n} b_k^{1/p}\right)^p \leq \left(\sum_{k=1}^{n}(a_k + b_k)^{1/p}\right)^p. \tag{4}$$

For $0 < p < 1$, *we have the converse inequalities in* (3) *and* (4).

If $a_k > 0$ and $b_k > 0$, for $k = 1, \ldots, n$, and $p < 0$, then (3) *holds with the sense of inequality changed.*

(3) is, in fact, the MINKOWSKI inequality. The case $0 < p < 1$ concerning this inequality is also considered in § 22 of the book [2] of E. F. BECKENBACH and R. BELLMAN.

For complex sequences the MINKOWSKI inequality reads:

Theorem 3. *If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are complex sequences, and if $p > 1$, then*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}.$$

Consider the set function M_p defined by

$$M_p(a, b; I) = \left(\left(\sum_{k\in I} |a_k|^p\right)^{1/p} + \left(\sum_{k\in I} |b_k|^p\right)^{1/p}\right)^p - \sum_{k\in I} |a_k + b_k|^p,$$

where I is an arbitrary nonempty finite subset of the set of positive integers and a_k and b_k for $k \in I$ denote complex numbers.

H. W. MCLAUGHLIN and F. T. METCALF [3] proved the following result involving M_p:

Theorem 4. *Let I and J denote nonempty disjoint finite sets of distinct positive integers. Suppose that (a_k) and (b_k), with $k \in I \cup J$, are sequences of complex numbers. If $p > 1$ or $p < 0$ (in the second case it will be assumed that a_k, b_k and $a_k + b_k$ are nonzero), then*

$$M_p(a, b; I \cup J) \geq M_p(a, b; I) + M_p(a, b; J), \tag{5}$$

where equality holds if and only if the ordered pairs

$$\left(\sum_{k\in I} |a_k|^p, \sum_{k\in I} |b_k|^p\right) \quad \textit{and} \quad \left(\sum_{k\in J} |a_k|^p, \sum_{k\in J} |b_k|^p\right)$$

are proportional. If $0 < p < 1$, then the sense of this inequality reverses, while the necessary and sufficient condition for equality remains unchanged. If $p = 1$, then equality always holds in (5).

The above result, in fact, extends inequality (1) of MINKOWSKI.

Replacing M_p by the following function of the index set I

$$\frac{\left(\sum_{k\in I} |a_k + b_k|^p\right)^{1/p}}{\left(\sum_{k\in I} |a_k|^p\right)^{1/p} + \left(\sum_{k\in I} |b_k|^p\right)^{1/p}},$$

similar results were obtained in [3].

The following result of H. P. MULHOLLAND [4] presents also a generalization of MINKOWSKI's inequality (1).

Theorem 5. *Let the function f for $x \geq 0$ be increasing and convex with $f(0) = 0$. Furthermore, let the function F, defined by $F(t) = \log f(e^t)$, be* *convex for all real t. Then for the sequences of nonnegative numbers* $a_1, \ldots, a_n$, *and* $b_1, \ldots, b_n$,

$$f^{-1}\left(\sum_{k=1}^{n} f(a_k + b_k)\right) \leq f^{-1}\left(\sum_{k=1}^{n} f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n} f(b_k)\right).$$

In the same paper there are other inequalities which are connected with MINKOWSKI's inequality (1).

An integral analogue to inequality (1) is given by

Theorem 6. *Let f and g be real-valued functions defined on* $[a, b]$ *such that the functions* $x \mapsto |f(x)|^p$ *and* $x \mapsto |g(x)|^p$, *for* $p > 1$, *are integrable on* $[a, b]$. *Then*

$$\left(\int_a^b |f(x) + g(x)|^p\, dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p\, dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\, dx\right)^{1/p}.$$

Equality holds if and only if $f(x) = 0$ *almost everywhere, or* $g(x) = \alpha f(x)$ *almost everywhere with a constant* $\alpha \geq 0$.

Concerning Theorem 6, see, for example, the book [5] of N. I. ACHIESER, pp. 4—7.

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2.10 Inequalities of Aczél, Popoviciu, Kurepa and Bellman

Theorem 1. *Let* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *be two sequences of real numbers, such that*

$$b_1^2 - b_2^2 - \cdots - b_n^2 > 0, \text{ or } a_1^2 - a_2^2 - \cdots - a_n^2 > 0.$$

Then

$$(a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2) \leq (a_1 b_1 - a_2 b_2 - \cdots - a_n b_n)^2, \tag{1}$$

with equality if and only if the sequences a and b are proportional.

This is ACZÉL's inequality [1].

Proof. First, we assume that a and b are not proportional, and let $b_1^2 - b_2^2 - \cdots - b_n^2 > 0$. We have

$$(2) \quad f(x) = (b_1^2 - b_2^2 - \cdots - b_n^2)\, x^2 - 2(a_1 b_2 - a_2 b_2 - \cdots - a_n b_n)\, x + (a_1^2 - a_2^2 - \cdots - a_n^2)$$
$$= (b_1 x - a_1)^2 - (b_2 x - a_2)^2 - \cdots - (b_n x - a_n)^2.$$

Again, by the above hypotheses, we have $b_1 \neq 0$ and

$$f\left(\frac{a_1}{b_1}\right) = -\left(b_2 \frac{a_1}{b_1} - a_2\right)^2 - \cdots - \left(b_n \frac{a_1}{b_1} - a_n\right)^2 < 0.$$

Since $f(x) \to +\infty$ $(x \to +\infty)$ and $f(x) \to +\infty$ $(x \to -\infty)$, we conclude that the polynomial f has two zeros belonging to the intervals $\left(-\infty, \frac{a_1}{b_1}\right)$ and $\left(\frac{a_1}{b_1}, +\infty\right)$.

Therefore, since the discriminant of the polynomial (2) must be positive, we obtain (1).

From (2) it follows that there is equality in (1) if and only if a and b are proportional.

Owing to the symmetry, the proof is the same if the condition $b_1^2 - b_2^2 - \cdots - b_n^2 > 0$ is replaced by $a_1^2 - a_2^2 - \cdots - a_n^2 > 0$.

The following generalization of (1) is due to T. Popoviciu [2]:

Theorem 2. *If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are sequences of non-negative real numbers such that*

$$(3) \qquad a_1^p - a_2^p - \cdots - a_n^p > 0, \quad \text{or} \quad b_1^p - b_2^p - \cdots - b_n^p > 0,$$

then, for $p \geq 1$,

$$(a_1^p - a_2^p - \cdots - a_n^p)(b_1^p - b_2^p - \cdots - b_n^p) \leq (a_1 b_1 - a_2 b_2 - \cdots - a_n b_n)^p.$$

R. Bellman [3] proved the following result:

Theorem 3. *If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are sequences of non-negative real numbers which satisfy (3), then, for $p > 1$,*

$$(a_1^p - a_2^p - \cdots - a_n^p)^{1/p} + (b_1^p - b_2^p - \cdots - b_n^p)^{1/p}$$
$$\leq \left((a_1 + b_1)^p - (a_2 + b_2)^p - \cdots - (a_n + b_n)^p\right)^{1/p}.$$

In the mentioned paper [2], T. Popoviciu gave simple proofs of Theorems 1, 2 and 3.

The following generalization of (1) is due to S. Kurepa [4]:

Theorem 4. *Let B be a Hermitian functional on a vector space X. Set*

$$X_0 = \{x \in X \mid B(x, x) = 0\}, \quad K = \{x \in X \mid B(x, x) < 0\}$$

and for a given vector $z \in X$, $X^{(z)} = \{x \in X \mid B(x, z) = 0\}$.

1° *If $e \in X$ is such that $B(e, e) \neq 0$, then $X = L(e) \oplus X^{(e)}$ where $\oplus$ denotes the direct sum of spaces $X^{(e)}$ and $L(e)$ is a subspace spanned by e.*

2° *If the functional B is positive semidefinite on $X^{(e)}$ for at least one $e \in K$, then B is positive semidefinite on $X^{(f)}$ for every $f \in K$.*

3° *The functional B is positive semidefinite on $X^{(e)}$ $(e \in K)$ if and only if the inequality*

$$|B(x, y)|^2 \geq B(x, x)\, B(y, y) \tag{4}$$

holds for every $x \in K$ and all $y \in X$. Equality holds in (4) if and only if there is a scalar a such that $y - ax \in X_0 \cap X^{(x)}$.

For other generalizations, see also [5] and [6]. In [6] inequality (1) is generalized to a Hilbert space and it is obtained by the use of the complexification of a real Hilbert space.

Remark. H. Schwerdtfeger in Canad. Math. Bull. **1**, 175—179 (1958) has obtained certain results on possible combinations of signs + and — which allow inequalities analogous to (1).

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2.11 Schweitzer's, Diaz-Metcalf's, Rennie's and Related Inequalities

P. Schweitzer [1] proved the inequality

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right) \leq \frac{(M+m)^2}{4Mm}, \tag{1}$$

where $0 < m \leq a_k \leq M$ for $k = 1, \ldots, n$.

In the same paper, P. Schweitzer has also shown that if functions $x \mapsto f(x)$ and $x \mapsto \frac{1}{f(x)}$ are integrable on $[a, b]$ and $0 < m \leq f(x) \leq M$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \int_a^b \frac{1}{f(x)}\,dx \leq \frac{(M+m)^2}{4Mm}(b-a)^2. \tag{2}$$

G. PÓLYA and G. SZEGÖ [2] proved that

$$\frac{\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \leq \left(\frac{\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}}{2}\right)^2, \tag{3}$$

where

$$0 < m_1 \leq a_k \leq M_1, \quad 0 < m_2 \leq b_k \leq M_2 \quad \text{for} \quad k = 1, \ldots, n. \tag{4}$$

Inequality (3) reduces to equality if and only if the numbers

$$K = \frac{\frac{M_1}{m_1}}{\frac{M_1}{m_1} + \frac{M_2}{m_2}} n, \quad L = \frac{\frac{M_2}{m_2}}{\frac{M_1}{m_1} + \frac{M_2}{m_2}} n$$

are positive integers and if K of the numbers $a_1, \ldots, a_n$ are equal to m_1 and L of these numbers equal to M_1, and if the corresponding numbers b_k are equal to M_2 and m_2 respectively.

L. B. KANTOROVIČ [3] proved that

$$\left(\sum_{k=1}^{n} \gamma_k u_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{\gamma_k} u_k^2\right) \leq \frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2 \left(\sum_{k=1}^{n} u_k^2\right)^2, \tag{5}$$

where $0 < m \leq \gamma_k \leq M$ for $k = 1, \ldots, n$, and he pointed out that inequality (5) is a particular case of (3).

W. GREUB and W. RHEINBOLDT [4] proved that

$$\left(\sum_{k=1}^{n} a_k^2 u_k^2\right)\left(\sum_{k=1}^{n} b_k^2 u_k^2\right) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{k=1}^{n} a_k b_k u_k^2\right)^2, \tag{6}$$

where $0 < m_1 \leq a_k \leq M_1$ and $0 < m_2 \leq b_k \leq M_2$ for $k = 1, \ldots, n$.

Remark 1. The integral analogues of (3), (5) and (6) are known and are similar to the analogue (2) of (1).

Remark 2. Inequalities (3), (5) and (6) can be deduced from (2). Indeed, putting in (2) $a = 0$, $b = a_1 b_1 + \cdots + a_n b_n$ and

$$f(x) = \frac{a_1}{b_1} \text{ for } 0 \leq x < a_1 b_1 \text{ and } f(x) = \frac{a_k}{b_k} \text{ for } \sum_{r=1}^{k-1} a_r b_r \leq x < \sum_{r=1}^{k} a_r b_r \ (k = 2, \ldots, n),$$

where

$$0 < m_1 \leq a_k \leq M_1 \quad \text{and} \quad 0 < m_2 \leq b_k \leq M_2 \quad \text{for} \quad k = 1, \ldots, n,$$

we have

$$m = \frac{m_1}{M_2} < \frac{a_k}{b_k} < \frac{M_1}{m_2} = M,$$

and this reduces (2) to (3).

For $a = 0$, $b = u_1^2 + \cdots + u_n^2$ and

$$f(x) = \gamma_1 \text{ for } 0 \leq x < u_1^2 \text{ and } f(x) = \gamma_k \text{ for } \sum_{r=1}^{k-1} u_r^2 \leq x < \sum_{r=1}^{k} u_r^2 \ (k = 2, \ldots, n),$$

where

$$0 < m \leq \gamma_k \leq M \text{ for } k = 1, \ldots, n,$$

from (2) we obtain (5).

Finally, if we replace γ_k by a_k/b_k and u_k by $\sqrt{a_k b_k}\, u_k$, where

$$0 < m_1 \leq a_k \leq M_1, \text{ and } 0 < m_2 \leq b_k \leq M_2 \text{ for } k = 1, \ldots, n,$$

we have

$$m = \frac{m_1}{M_2} \leq \gamma_k \leq \frac{M_1}{m_2} = M,$$

and this reduces (5) to (6).

Applying the above procedure, E. Makai [5] in 1961 showed that the inequality of Kantorovič can be deduced from (2). At the same time, P. Henrici [6] also gave an easy dérivation of (5) from (1).

Conversely, it is clear that (6) implies inequalities (1), (3) and (5).

The above simple results have evoked, in the recent time, a considerable interest, and many refinements, generalizations and inversions of cited inequalities, together with various applications, have been given. Concerning this, see, in particular, papers [7]—[24].

J. B. Diaz and F. T. Metcalf [14] proved

Theorem 1. *If a_k ($\neq 0$) and b_k ($k = 1, \ldots, n$) are real numbers and if*

$$m \leq \frac{b_k}{a_k} \leq M \quad \textit{for} \quad k = 1, \ldots, n, \tag{7}$$

then

$$\sum_{k=1}^{n} b_k^2 + mM \sum_{k=1}^{n} a_k^2 \leq (M + m) \sum_{k=1}^{n} a_k b_k. \tag{8}$$

Equality holds in (8) *if and only if in each of the n inequalities* (7) *at least one equality sign holds, i.e., either $b_k = m a_k$ or $b_k = M a_k$ (where the equation may vary with k).*

The proof of Theorem 1 is easy and elementary. Indeed, by virtue of (7), the inequality

$$0 \leq \left(\frac{b_k}{a_k} - m\right)\left(M - \frac{b_k}{a_k}\right) a_k^2$$

is obvious. Summing up, we get

$$0 \leq \sum_{k=1}^{n} (b_k - m a_k)(M a_k - b_k), \tag{9}$$

i.e.,

$$0 \leq -\sum_{k=1}^{n} [b_k^2 - (M + m) a_k b_k + M m a_k^2],$$

which is inequality (8).

In inequality (9), or equivalently, in inequality (8), equality holds if and only if each of the summands $(b_k - ma_k)(Ma_k - b_k)$ is equal to zero, which proves Theorem 1.

Inequalities (1), (3), (5) and (6) are contained in inequality (8) as special cases, as well as some other inequalities. Let us prove it for inequality (3).

Putting $m = m_2/M_1$, $M = M_2/m_1$ in (8), we obtain

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^{n} a_k b_k .$$

Adding to the last inequality the following obvious inequality

$$0 \leq \left[\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} - \left(\frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} a_k^2\right)^{1/2}\right]^2,$$

we find

$$0 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^{n} a_k b_k - 2\left(\frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2\right)^{1/2},$$

i.e.,

$$\frac{\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \leq \frac{1}{4}\,\frac{M_1 m_1}{M_2 m_2}\left(\frac{M_1 M_2 + m_1 m_2}{M_1 m_1}\right)^2 .$$

Since

$$\frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 = \frac{1}{4}\,\frac{M_1 m_1}{M_2 m_2}\left(\frac{M_1 M_2 + m_1 m_2}{M_1 m_1}\right)^2,$$

we have already proved that (3) is a consequence of (8).

Consider now $M_n^{[r]}(a;p) = \left(\sum_{k=1}^{n} p_k a_k^r\right)^{1/r}$ with $\sum_{k=1}^{n} p_k = 1$. W. Specht (see 2.14.3) proved that

$$\frac{M_n^{[s]}(a;p)}{M_n^{[r]}(a;p)} \leq \left(\frac{r}{q^r - 1}\right)^{1/s}\left(\frac{q^s - 1}{s}\right)^{1/r}\left(\frac{q^s - q^r}{s - r}\right)^{1/s - 1/r}, \tag{10}$$

where $0 < m \leq a_k \leq M$ $(k = 1, \ldots, n)$, $q = M/m$, $s > r$ and $sr \neq 0$. For $s = 1$ and $r = -1$ the above inequality is, in fact, (5).

A. J. Goldman [19] deduced inequality (10) from the following result: If $sr < 0$, then

$$(M^s - m^s)\, M_n^{[r]}(a;p)^r - (M^r - m^r)\, M_n^{[s]}(a;p)^s \leq M^s m^r - M^r m^s . \tag{11}$$

The opposite inequality holds for $sr > 0$.

Inequality (11) was also proved by A. W. Marshall and I. Olkin [21], and B. C. Rennie [12].

For $s = 1$ and $r = -1$ inequality (11) yields the following inequality of B. C. RENNIE [12]:

$$\sum_{k=1}^{n} p_k a_k + mM \sum_{k=1}^{n} \frac{p_k}{a_k} \leq m + M.$$

J. B. DIAZ, A. J. GOLDMAN and F. T. METCALF [20] showed that the inequality which appears in Theorem 1 is equivalent to that of B. C. RENNIE.

J. B. DIAZ and F. T. METCALF in [14] also proved the following two theorems:

Theorem 2. *Let $a_k \neq 0$ and b_k $(k = 1, \ldots, n)$ be complex numbers such that*

$$m \leq \operatorname{Re}\frac{b_k}{a_k} + \operatorname{Im}\frac{b_k}{a_k} \leq M, \tag{12}$$

$$m \leq \operatorname{Re}\frac{b_k}{a_k} - \operatorname{Im}\frac{b_k}{a_k} \leq M \tag{13}$$

for $k = 1, \ldots, n$. Then

$$\sum_{k=1}^{n} |b_k|^2 + mM \sum_{k=1}^{n} |a_k|^2 \leq (M + m) \operatorname{Re} \sum_{k=1}^{n} a_k \overline{b_k} \leq |M + m| \left| \sum_{k=1}^{n} a_k \overline{b_k} \right|. \tag{14}$$

Equality holds on the left of (14) *if and only if for each k such that* $\operatorname{Im}(b_k/a_k) \neq 0$ *one equality sign holds in* (12) *and one equality sign (necessarily the "opposite" one) holds in* (13); *while for each k such that* $\operatorname{Im}(b_k/a_k) = 0$, *at least one of the equality signs holds in* (12) (*or, what is the same in this case, in* (13)).

Theorem 3. *Let f and g be real-valued and square integrable functions on* $[a, b]$. *Assume that*

$$m \leq \frac{g(x)}{f(x)} \leq M \quad (\textit{with } f(x) \neq 0) \tag{15}$$

for almost every $x \in [a, b]$. Then

$$\int_a^b g(x)^2\, dx + Mm \int_a^b f(x)^2\, dx \leq (M + m) \int_a^b f(x)\, g(x)\, dx. \tag{16}$$

Equality holds in (16) *if and only if, for almost every x in* $[a, b]$, *at least one of the equality signs holds in* (15), *where the equality sign in question may vary with x.*

Inequality (16) taken together with the obvious inequality

$$0 \leq \left[\left(\int_a^b g(x)^2\, dx\right)^{1/2} - \left(Mm \int_a^b f(x)^2\, dx\right)^{1/2}\right]^2$$

gives some integral inequalities proved by G. PÓLYA, P. SCHWEITZER and J. KÜRSCHAK (see [2]).

Among some other results proved by J. B. DIAZ and F. T. METCALF we mention the following theorem which is more general than Theorem 1.

Theorem 4. *Let the complex numbers a_k ($\neq 0$), b_k ($k = 1, \ldots, n$), m and M satisfy*

$$\operatorname{Re} m + \operatorname{Im} m \leq \operatorname{Re} \frac{b_k}{a_k} + \operatorname{Im} \frac{b_k}{a_k} \leq \operatorname{Re} M + \operatorname{Im} M,$$

$$\operatorname{Re} m - \operatorname{Im} m \leq \operatorname{Re} \frac{b_k}{a_k} - \operatorname{Im} \frac{b_k}{a_k} \leq \operatorname{Re} M - \operatorname{Im} M,$$

for $k = 1, \ldots, n$. Then

$$\sum_{k=1}^{n} |b_k|^2 + \left(\operatorname{Re}(m\overline{M})\right) \sum_{k=1}^{n} |a_k|^2$$
$$\leq \operatorname{Re}\left((M + m) \sum_{k=1}^{n} a_k \overline{b_k}\right) \leq |M + m| \left|\sum_{k=1}^{n} a_k \overline{b_k}\right|.$$

Inequality (3) yields an upper bound of the following quotient

$$\frac{\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)}{\left(\sum_{k=1}^{n} a_k b_k\right)^2},$$

while a lower bound is given by CAUCHY's inequality (see 2.6.1). Therefore, inequality (3) and its analogues are called in the literature inverse or complementary to the CAUCHY inequality.

There exist also the inverse inequalities of the other important inequalities. So, for instance, the GRÜSS inequality (see 2.13) is the inverse of ČEBYŠEV's inequality. We shall mention the inverse of HÖLDER's inequality, proved by J. B. DIAZ, A. J. GOLDMAN and F. T. METCALF [20].

Theorem 5. *Let functions $x \mapsto f(x)^p$ and $x \mapsto g(x)^q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$ be positive and integrable on $[a, b]$ and let, on $[a, b]$,*

$$0 < m_1 \leq f(x) \leq M_1 < +\infty, \quad 0 < m_2 \leq g(x) \leq M_2 < +\infty.$$

Then

$$\left(\int_a^b f(x)^p \, dx\right)^{1/p} \left(\int_a^b g(x)^q \, dx\right)^{1/q} \leq C_p \int_a^b f(x)\, g(x)\, dx,$$

where

$$C_p = \frac{M_1^p M_2^q - m_1^p m_2^q}{\left(p m_2 M_2 (M_1 M_2^{q-1} - m_1 m_2^{q-1})\right)^{1/p} \left(q m_1 M_1 (M_2 M_1^{p-1} - m_2 m_1^{p-1})\right)^{1/q}}.$$

Z. NEHARI [24] deduced many generalizations of the inequalities inverse to HÖLDER's inequality. We mention only the following:

Theorem 6. *Let $f_1, \ldots, f_n$ be real-valued nonnegative concave functions on a real interval $[a, b]$. If $p_k > 0$ for $k = 1, \ldots, n$ and $p_1^{-1} + \cdots + p_n^{-1} = 1$, then*

$$\prod_{k=1}^{n} \left(\int_a^b f_k(x)^{p_k} dx \right)^{p_k^{-1}} \leq C_n \int_a^b \left(\prod_{k=1}^{n} f_k(x) \right) dx,$$

where

$$C_n = \frac{(n+1)!}{\left(\left[\frac{n}{2}\right]!\right)^2 \prod_{k=1}^{n} (p_k + 1)^{1/p_k}}.$$

Equality holds if $f_k(x) = x$ for $\left[\frac{n}{2}\right]$ of the subscripts k, and $f_k(x) = 1 - x$ otherwise.

A number of papers had as their aim to extend the aforementioned inequalities in the HILBERT, BANACH or other spaces. Concerning this, see, for example, paper [17].

Remark 3. In connection with the above topics, it is of interest to consult the review of papers [15], [16] and [17] by S. D. CHATTERJI published in Zentralblatt für Mathematik **135**, **347** (**1967**).

Remark 4. Concerning paper [24], see also a review of J. V. RYFF in Math. Reviews **37**, **289** (**1969**).

Remark 5. For some interesting generalizations of the KANTOROVIČ inequality, see paper [25] of E. BECK.

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2.12 An Inequality of Fan and Todd

In the book [1] of A. M. OSTROWSKI the following result is noted:

Theorem 1. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences, not proportional, of real numbers. Let $x = (x_1, \ldots, x_n)$ be any sequence of real numbers for which the following holds*

$$(1) \qquad \sum_{i=1}^{n} a_i x_i = 0, \quad \sum_{i=1}^{n} b_i x_i = 1.$$

Then

(2)
$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2},$$

with equality if and only if

(3)
$$x_k = \frac{b_k \sum_{i=1}^{n} a_i - a_k \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \quad \textit{for} \quad k = 1, \ldots, n.$$

Proof. Let

(4)
$$A = \sum_{i=1}^{n} a_i^2, \ B = \sum_{i=1}^{n} b_i^2, \ C = \sum_{i=1}^{n} a_i b_i \quad \text{and} \quad y_i = \frac{Ab_i - Ca_i}{AB - C^2}.$$

It can be verified that the sequence $y_1, \ldots, y_n$ satisfies (1). Any sequence $x_1, \ldots, x_n$ subject to (1) satisfies

$$\sum_{i=1}^{n} x_i y_i = \frac{A}{AB - C^2},$$

and, in particular, we have

$$\sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2}.$$

Hence, any sequence $x_1, \ldots, x_n$ subject to (1) satisfies

(5)
$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (x_i - y_i)^2,$$

wherefrom follows

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2},$$

which was to be proved.

From (5) it follows that equality holds in the above inequality if and only if $x_i = y_i$ for $i = 1, \ldots, n$, i.e., if and only if condition (3) is fulfilled.

Remark 1. The case of equality is not stated in the book [1].

K. Fan and J. Todd [2] proved:

Theorem 2. *Let* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ $(n \geq 2)$ *be two sequences of real numbers such that* $a_i b_j \neq a_j b_i$ *for* $i \neq j$.

Then

$$\frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \binom{n}{2}^{-2} \sum_{i=1}^{n} \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{a_j}{a_j b_i - a_i b_j}\right)^2.$$

Proof. Let

$$x_i = \binom{n}{2}^{-1} \sum_{j\neq i} \frac{a_j}{a_j b_i - a_i b_j} \quad \text{for} \quad 1 \le i \le n.$$

In the double sum

$$\sum_{i=1}^{n} a_i x_i = \binom{n}{2}^{-1} \sum_{i} \sum_{j\neq i} \frac{a_i a_j}{a_j b_i - a_j b_i}$$

the $n(n-1)$ terms can be grouped in pairs of the form

$$\binom{n}{2}^{-1} \left(\frac{a_i a_j}{a_j b_i - a_i b_j} + \frac{a_j a_i}{a_i b_j - a_j b_i}\right) \quad (i \neq j),$$

and the sum of each such pair vanishes.

Hence, we deduce that $\sum_{i=1}^{n} a_i x_i = 0$.

Similarly, we find that $\sum_{i=1}^{n} b_i x_i = 1$.

According to Theorem 1, we derive

$$\sum_{i=1}^{n} x_i^2 \ge \frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2},$$

which was to be proved.

Remark 2. If $a_i = \sin \alpha_i$ and $b_i = \cos \alpha_i$ for $i = 1, \ldots, n$, one obtains an inequality proved by J. B. Chassan [3] using arguments based on statistical considerations.

K. Fan and J. Todd in [2] also proved the following generalization of Theorem 2:

Theorem 3. *Let p_{ij} $(i, j = 1, \ldots, n;\ i \neq j)$ be real numbers such that*

$$p_{ij} = p_{ji}, \quad P_{ij} = \sum_{1 \le i < j \le n} p_{ij} \neq 0.$$

Then for any two sequences of real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfying $a_i b_j \neq a_j b_i$ $(i \neq j)$, we have

$$\frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \frac{1}{P^2} \sum_{i=1}^{n} \left(\sum_{j\neq i} \frac{p_{ij} a_j}{a_j b_i - a_i b_j}\right)^2.$$

For complex numbers we have the corresponding result:

Theorem 4. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of complex numbers which are not proportional. If for a complex sequence $x = (x_1, \ldots, x_n)$ the following conditions hold*

$$\sum_{i=1}^{n} a_i \overline{x_i} = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i \overline{x_i} = 1,$$

then

$$\sum_{i=1}^{n} |x_i|^2 \geq \frac{\sum_{i=1}^{n} |a_i|^2}{\left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right) - \left|\sum_{i=1}^{n} a_i \overline{b_i}\right|^2}.$$

The following generalization of Theorem 2 is also due to K. FAN and J. TODD [2]:

Theorem 5. *Let m and n be integers such that $2 \leq m \leq n$. Let*

$$x_i = (x_{i1}, \ldots, x_{in}) \quad \text{for} \quad 1 \leq i \leq m$$

be m vectors in the unitary space X such that every $m \times m$ submatrix of the $m \times n$ matrix

$$\left\| \begin{matrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & & & \\ x_{m1} & x_{m2} & & x_{mn} \end{matrix} \right\| \tag{6}$$

is nonsingular.

Let $M(j_1, \ldots, j_{m-1})$ denote the determinant of order $m - 1$ formed by the first $m - 1$ rows of (6) *and the columns of* (6) *with indices $j_1, \ldots, j_{m-1}$ taken in this order. Let $N(j_1, \ldots, j_m)$ denote the determinant of order m formed by the columns of* (6) *with indices $j_1, \ldots, j_m$ taken in this order. Then*

$$\frac{\Gamma(x_1, \ldots, x_{m-1})}{\Gamma(x_1, \ldots, x_m)} \leq \binom{n}{m}^{-2} \sum_{j_m=1}^{n} \left| \sum_{\substack{j_1 < \cdots < j_{m-1} \\ j_1, \ldots, j_{m-1} \neq j_m}} \frac{M(j_1, \ldots, j_{m-1})}{N(j_1, \ldots, j_m)} \right|^2, \tag{7}$$

where Γ is GRAM'S *determinant.*

The summation inside the absolute value sign of (7) *sums over all $(m - 1)$-tuples $(j_1, \ldots, j_{m-1})$ of integers different from the fixed j_m and such that $1 \leq j_1 < \cdots < j_{m-1} \leq n$.*

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2.13 Grüss' Inequality

Let f and g be two functions defined and integrable over (a, b). Let

$$\varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma$$

for all $x \in (a, b)$, where φ, Φ, γ, Γ are fixed real constants.

H. GRÜSS (see footnote in [1]) conjectured that

$$(1) \qquad \left| \frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \frac{1}{(b-a)^2} \int_a^b f(x)\, dx \int_a^b g(x)\, dx \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).$$

G. GRÜSS [1] proved that (1) is true and that the constant 1/4 is the best possible.

We shall give a proof of (1). First, by making the substitution $x = (t-a)/(b-a)$ the problem is reduced to the special case $a = 0$, $b = 1$. In that case we write

$$F = \int_0^1 f(x)\, dx, \quad G = \int_0^1 g(x)\, dx,$$

and

$$D(f, g) = \int_0^1 f(x)\, g(x)\, dx - FG.$$

Then (1) reads

$$(2) \qquad |D(f, g)| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).$$

Note that

$$(3) \qquad D(f, f) = \int_0^1 f(x)^2\, dx - \left(\int_0^1 f(x)\, dx \right)^2 \geq 0$$

holds by the BUNIAKOWSKI-SCHWARZ inequality.

On the other hand,

$$D(f, f) = (\Phi - F)(F - \varphi) - \int_0^1 [\Phi - f(x)]\, [f(x) - \varphi]\, dx,$$

which implies that

$$(4) \qquad D(f, f) \leq (\Phi - F)(F - \varphi).$$

One can easily verify that

$$D(f, g) \leq \int_0^1 [f(x) - F]\, [g(x) - G]\, dx.$$

Using the BUNIAKOWSKI-SCHWARZ inequality, we get

$$D(f, g)^2 \leq \int_0^1 [f(x) - F]^2\, dx \int_0^1 [g(x) - G]^2\, dx = D(f, f)\, D(g, g).$$

According to (3) and (4), we infer that

$$D(f, g)^2 \leq (\Phi - F)(F - \varphi)(\Gamma - G)(G - \gamma). \tag{5}$$

Since

$$4(\Phi - F)(F - \varphi) \leq (\Phi - \varphi)^2,$$
$$4(\Gamma - G)(G - \gamma) \leq (\Gamma - \gamma)^2,$$

we conclude that (5) implies (2).

Taking $f(x) = g(x) = \operatorname{sgn}(2x - 1)$, the constant $1/4$ in (2) is seen to be the best possible.

Remark. A. M. OSTROWSKI has given some interesting results which are connected with inequality (1). See Basel Math. Notes No. **23** and **24** (1968), and Aequationes Math. **2**, 362–363 (1969).

Inequality (2) can be improved if we impose further restrictions on f and g. Let $\Delta_h^n f(x)$ be the n-th difference, i.e.,

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh).$$

A function f, defined over (a, b), is said to be monotonic of order p if

$$\Delta_h^n f(x) \geq 0, \quad \text{or} \quad \Delta_h^n f(x) \leq 0$$

for all $n = 1, \ldots, p$ and all $x \in (a, b)$, $h > 0$, $x + nh < b$. If f is monotonic of order p in (a, b) for all $p = 1, 2, \ldots$, we say that it is absolutely monotonic in (a, b) ("vollmonoton", "absolument monotone").

S. BERNSTEIN [2] proved that a function f monotonic of order $p \geq 2$ over (a, b) has continuous derivatives $f'(x), \ldots, f^{(p-1)}(x)$ and $f^{(p-1)}(x)$ has right and left derivatives for all $x \in (a, b)$. Consequently, if f is absolutely monotonic in (a, b), then it has derivatives of all orders and

$$f^{(n)}(x) \geq 0, \quad \text{or} \quad f^{(n)}(x) \leq 0$$

for all $n = 1, 2, \ldots$ and $x \in (a, b)$.

If f and g are absolutely monotonic functions in $(0, 1)$, G. GRÜSS [1] proved that

$$D(f, f)^2 \leq \frac{4}{45} (\Phi - \varphi)^2, \tag{6}$$

and

$$|D(f, g)| \leq \frac{4}{45} (\Phi - \varphi)(\Gamma - \gamma). \tag{7}$$

The constant 4/45 in (6), and also in (7), is the best possible, as is seen by taking $f(x) = x^2$.

G. Grüss in his proof of (6) and (7) used the Bernstein polynomials. A simpler proof was given by E. Landau [3]. He derived (6) and (7) from the following proposition:

If n is a positive integer and

$$A_r = \sum_{k=0}^{r} \binom{r}{k} p_k \quad \text{with} \quad p_k \geq 0 \ (r = 0, 1, \ldots, n),$$

then

$$\frac{1}{n}\sum_{r=0}^{n-1} A_r^2 - \frac{1}{n^2}\left(\sum_{r=0}^{n} A_r\right)^2 \leq \frac{4}{45}(A_n - A_0)^2.$$

E. Landau [4] proved that (6) and (7) hold if f and g are monotonic of order 4. Needless to say this is a nontrivial improvement of Grüss' result. For functions monotonic of order $k = 1, 2, 3$ E. Landau proved

$$|D(f, g)| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) \qquad \text{for} \quad k = 1,$$

$$|D(f, g)| \leq \frac{1}{9}(\Phi - \varphi)(\Gamma - \gamma) \qquad \text{for} \quad k = 2,$$

$$|D(f, g)| \leq \frac{9}{100}(\Phi - \varphi)(\Gamma - \gamma) \qquad \text{for} \quad k = 3.$$

G. H. Hardy [5] established an improvement of (7) for a more restrictive class of functions. Following his definition, we shall say that f is totally monotonic on (a, b), if

$$f(x) \geq 0, \quad f'(x) \leq 0, \quad f''(x) \geq 0, \quad f'''(x) \leq 0, \ldots$$

in (a, b). We note that in this case $f(-x)$ is absolutely monotonic in $(-b, -a)$.

Hardy's result is: If f and g are totally monotonic on $(0, +\infty)$ and $\varphi \leq f(x) \leq \Phi$, $\gamma \leq g(x) \leq \Gamma$ on $(0, a)$, then

$$|D(f, g)| \leq \frac{1}{12}(\Phi - \varphi)(\Gamma - \gamma), \tag{8}$$

where

$$D(f, g) = \frac{1}{a}\int_0^a f(x)\, g(x)\, dx - \frac{1}{a^2}\int_0^a f(x)\, dx \int_0^a g(x)\, dx.$$

The constant 1/12 is the best possible.

For example, we cannot apply (8) to $f(x) = (a - x)^2$ $(0 \leq x \leq a)$, since this function is not a restriction of a function totally monotonic on $(0, +\infty)$.

The estimates obtained so far for $|D(f, g)|$ involve only the bounds φ, Φ, γ, Γ. There is another kind of estimate which include also F and G, for instance (5). From a result of PH. FRANCK and G. PICK [6] the following theorem can be derived (cf. also [1]).

If f and g are positive convex functions in $(0, 1)$, then

$$|D(f, g)| \leq \frac{1}{3} F \cdot G.$$

Some other estimates can be derived from theorems proved by PH. FRANCK and G. PICK [6], and G. PICK and W. BLASCHKE [7].

Further estimates of this type were obtained by M. BIERNACKI, H. PIDEK and C. RYLL-NARDZEWSKI [8], and S. FEMPL [9].

In addition, K. KNOPP [10] has generalized GRÜSS' inequality to other means.

GRÜSS' inequality provides the bounds for the difference $D(f, g)$. An analogous result for the ratio

$$R(f, g) = \frac{\int_0^1 f(x)\, dx \int_0^1 g(x)\, dx}{\int_0^1 f(x)\, g(x)\, dx}$$

was obtained by J. KARAMATA [11]. He, namely, proved that if f and g are integrable functions on $(0, 1)$ and if

$$0 < a \leq f(x) \leq A \quad \text{and} \quad 0 < b \leq g(x) \leq B \text{ for } 0 \leq x \leq 1,$$

then

$$\frac{1}{K^2} \leq R(f, g) \leq K^2, \quad \text{with} \quad K = \frac{\sqrt{ab} + \sqrt{AB}}{\sqrt{aB} + \sqrt{Ab}} \geq 1.$$

J. KARAMATA obtained this result as a consequence of a more general result for which A. BILIMOVIĆ, as an addition to the same paper [11], gave a simple geometric proof.

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2.14 Means

2.14.1 Definitions

Arithmetic, geometric and harmonic means are only special cases of more general means which we shall define now.

Definition 1. *For a sequence of positive numbers $a = (a_1, \ldots, a_n)$ and positive weights $p = (p_1, \ldots, p_n)$, the weighted mean of order r (r is an extended real number) is defined as*

$$
\begin{aligned}
(1) \qquad M_n^{[r]}(a; p) &= \left(\frac{\sum_{i=1}^{n} p_i a_i^r}{P_n}\right)^{1/r} && (r \neq 0, |r| < +\infty),\\
&= \left(\prod_{i=1}^{n} a_i^{p_i}\right)^{1/P_n} && (r = 0),\\
&= \min(a_1, \ldots, a_n) && (r = -\infty),\\
&= \max(a_1, \ldots, a_n) && (r = +\infty),
\end{aligned}
$$

where $P_n = \sum_{i=1}^{n} p_i$.

For $r = -1$ we obtain weighted harmonic, for $r = 0$ weighted geometric, and for $r = 1$ weighted arithmetic means.

The above defined mean is a continuous function with respect to r on the extended set of real numbers, since

$$
\begin{aligned}
\lim_{r\to 0} M_n^{[r]}(a; p) &= \left(\prod_{i=1}^{n} a_i^{p_i}\right)^{1/P_n},\\
\lim_{r\to +\infty} M_n^{[r]}(a; p) &= \max(a_1, \ldots, a_n),\\
\lim_{r\to -\infty} M_n^{[r]}(a; p) &= \min(a_1, \ldots, a_n).
\end{aligned}
$$

In applications normed means of order r are often considered instead of (1). Those are means of order r for numbers $a_1, \ldots, a_n$ with weights $p_1, \ldots, p_n$ such that $\sum_{i=1}^{n} p_i = 1$.

However, this is not an important limitation, since we can pass from ordinary weighted means to normed means by introducing new weights defined by

$$q_i = \frac{p_i}{p_1 + \cdots + p_n} \qquad (i = 1, \ldots, n).$$

Non-weighted means of order r (or simply: means of order r), i.e., those means for which $p_1 = \cdots = p_n = 1$, are of special interest. Therefore, as a special case of Definition 1, we have

Definition 2. *Let $a = (a_1, \ldots, a_n)$ be a finite sequence of positive numbers and let r be an extended real number. The mean of order r is defined by*

$$\text{(2)} \qquad \begin{aligned} M_n^{[r]}(a) &= \left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} && (r \neq 0,\ |r| < +\infty), \\ &= (a_1 \cdots a_n)^{1/n} && (r = 0), \\ &= \min(a_1, \ldots, a_n) && (r = -\infty), \\ &= \max(a_1, \ldots, a_n) && (r = +\infty). \end{aligned}$$

As can easily be seen, for $r = -1$ we obtain the harmonic mean, for $r = 0$ the geometric, and for $r = 1$ the arithmetic mean.

More general means than the ones defined by (1) are also considered in the literature. For example, there is the following definition:

Definition 3. *Let f be a real-valued function, which is monotone on the closed interval $[\alpha, \beta]$. If $a = (a_1, \ldots, a_n)$ is a sequence of real numbers from $[\alpha, \beta]$ and $p = (p_1, \ldots, p_n)$ a sequence of positive numbers, the quasi-arithmetic non-symmetrical mean of numbers $a_1, \ldots, a_n$ is defined by*

$$\text{(3)} \qquad M_f(a; p) = f^{-1}\left(\frac{p_1 f(a_1) + \cdots + p_n f(a_n)}{p_1 + \cdots + p_n}\right),$$

where f^{-1} is the inverse function of f.

If $p_1 = \cdots = p_n = 1$, we obtain symmetric quasi-arithmetic means of the numbers $a_1, \ldots, a_n$.

For $f(x) = x^r$ we get from (3) weighted means of order r.

For even more general means see [1].

The integral analogue of (1) is given by

Definition 4. *Let f and p be defined, positive and integrable functions on the closed interval $[a, b]$. The weighted mean of order r of the function f on*

$[a, b]$ with the weight p is

$$(4)\quad M^{[r]}(f; p; a, b) = \left(\frac{\int_a^b p(x)\, f(x)^r\, dx}{\int_a^b p(x)\, dx}\right)^{1/r} \qquad (r \neq 0,\ |r| < +\infty),$$

$$= \exp \frac{\int_a^b p(x) \log f(x)\, dx}{\int_a^b p(x)\, dx} \qquad (r = 0),$$

$$= m \qquad (r = -\infty),$$

$$= M \qquad (r = +\infty),$$

where $m = \inf f(x)$ and $M = \sup f(x)$ for $x \in [a, b]$.

For $p(x) = 1$ we have the integral analogue of (2).

More general means of functions are defined in the literature in a way similar to Definition 3.

Reference

1. MITRINOVIĆ, D. S., and P. M. VASIĆ: Sredine. Matematička Biblioteka, vol. **40**. Beograd 1969.

2.14.2 Inequalities Involving Means

Inequalities which hold for the arithmetic, geometric and harmonic means considered above, are special cases of the inequality which appears in the following

Theorem 1. *If $a_1 = \cdots = a_n = a_0$, then $M_n^{[r]}(a; p) = a_0$. Otherwise, $M_n^{[r]}(a; p)$ is a strictly increasing function of r, i.e., for $-\infty \leq s < t \leq +\infty$ we have*

$$(1)\qquad M_n^{[s]}(a; p) < M_n^{[t]}(a; p).$$

Proof. Consider the functions f and F defined by

$$f(t) = \left(\frac{p_1 a_1^t + \cdots + p_n a_n^t}{p_1 + \cdots + p_n}\right)^{1/t},$$

and

$$F(t) = t^2 \frac{f'(t)}{f(t)} = t^2 \frac{d}{dt}\left(\frac{1}{t} \log \frac{\sum_{i=1}^n p_i a_i^t}{P_n}\right)$$

$$= t\, \frac{\sum_{i=1}^n p_i a_i^t \log a_i}{\sum_{i=1}^n p_i a_i^t} - \log \frac{\sum_{i=1}^n p_i a_i^t}{P_n} \qquad \left(P_n = \sum_{i=1}^n p_i\right).$$

Since $f(t) > 0$, functions f' and F are of the same sign. Therefore, we should prove that $F(t) > 0$. Since

$$F'(t) = \frac{t}{\left(\sum_{i=1}^{n} p_i a_i^t\right)^2} \left(\left(\sum_{i=1}^{n} p_i a_i^t\right) \left(\sum_{i=1}^{n} p_i a_i^t \log^2 a_i\right) - \left(\sum_{i=1}^{n} p_i a_i^t \log a_i\right)^2 \right),$$

by Cauchy's inequality we conclude that F' has the same sign as t. Therefore, F is an increasing function for $t > 0$, decreasing for $t < 0$, and has a minimum for $t = 0$. Functions F and f' are, thus, positive for all values of t, except perhaps for $t = 0$. However, for $t = 0$,

$$f'(0) = \left(\prod_{i=1}^{n} a_i^{p_i}\right)^{1 / \sum_{i=1}^{n} p_i} \cdot \frac{\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i (\log a_i)^2 - \left(\sum_{i=1}^{n} p_i \log a_i\right)^2}{2P_n^2}$$

and $f'(0) = 0$ only in the case when $a_1 = \cdots = a_n$ (as the expression in the numerator of the above fraction is, according to Cauchy's inequality, always negative, except when the sequences $\sqrt{p_i} \log a_i$ and $\sqrt{p_i}$ are proportional).

Remark. Theorem 1 was proved in 1858 by O. Schlömilch [3] for the case $p_1 = \cdots = p_n$ whereas s and t are natural numbers, or belong to the set $\{1/2, 1/3, \ldots\}$.

For $p_1 = \cdots = p_n$ and arbitrary s and t, this theorem was proved in 1888 by H. Simon [4].

For arbitrary $p_1, \ldots, p_n$ Theorem 1 was formulated without proof by J. Bienaymé [5] in 1840. The first published proof is due to D. Besso [6].

The proof given here is a modification of the proof given by N. Norris [7] who used the above method for proving Theorem 1 for the case $p_1 = \cdots = p_n$.

We give the following theorem without proof:

Theorem 2. *Inequality*

$$M_f(a; p) \geq M_g(a; p)$$

holds in an interval I containing the numbers $a_1, \ldots, a_n$ if and only if:

1° *f is strictly increasing and g strictly decreasing in I,*

2° *f and g are strictly decreasing or strictly increasing in I, and g/f is strictly increasing in the same interval.*

In opposite cases inequality is reversed.

The proof of Theorem 2 can be found in the article [8] of R. Cooper (the theorem is proved there only for the case $p_1 = \cdots = p_n$, but the proof holds in the more general case, also).

Finally, notice that starting with these results, the corresponding theorems for the means defined by (4) in 2.14.1 can be obtained without difficulty.

More on inequalities involving means may be found in the books [1] and [2].

Using Theorem 1 we shall now prove

Theorem 3. *If* $\lambda_1 + \cdots + \lambda_n \geq 1$ *and* $\lambda_i > 0$, $A_{ji} > 0$, *then*

$$\sum_{i=1}^{m} A_{1i}^{\lambda_1} \cdots A_{ni}^{\lambda_n} \leq \left(\sum_{i=1}^{m} A_{1i}\right)^{\lambda_1} \cdots \left(\sum_{i=1}^{m} A_{ni}\right)^{\lambda_n}. \tag{2}$$

Proof. Let us prove this inequality for the case $\lambda_1 + \cdots + \lambda_n = 1$. Putting in (1)

$$s = 0, \quad r = 1, \quad a_j = \frac{A_{j\nu}}{\sum_{i=1}^{m} A_{ji}}, \quad p_j = \lambda_j \quad (j = 1, \ldots, n),$$

we get

$$\left(\frac{A_{1\nu}}{\sum_{i=1}^{m} A_{1i}}\right)^{\lambda_1} \cdots \left(\frac{A_{n\nu}}{\sum_{i=1}^{m} A_{ni}}\right)^{\lambda_n} \leq \lambda_1 \frac{A_{1\nu}}{\sum_{i=1}^{m} A_{1i}} + \cdots + \lambda_n \frac{A_{n\nu}}{\sum_{i=1}^{m} A_{ni}} \qquad (\nu = 1, \ldots, m).$$

Adding the last inequalities, according to the condition $\lambda_1 + \cdots + \lambda_n = 1$, we get (2).

Suppose now that $\lambda_1 + \cdots + \lambda_n = k > 1$. Putting $\lambda_i = k\lambda_i'$, where $k > 1$ and $\lambda_i' + \cdots + \lambda_n' = 1$ and $A_{ji}^k = A_{ji}'$, we find that

$$\begin{aligned} \sum_{i=1}^{m} A_{1i}^{\lambda_1} \cdots A_{ni}^{\lambda_n} &= \sum_{i=1}^{m} A_{1i}'^{\lambda_1'} \cdots A_{ni}'^{\lambda_n'} \leq \left(\sum_{i=1}^{m} A_{1i}'\right)^{\lambda_1'} \cdots \left(\sum_{i=1}^{m} A_{ni}'\right)^{\lambda_n'} \\ &= \left(\sum_{i=1}^{m} A_{1i}^k\right)^{\lambda_1/k} \cdots \left(\sum_{i=1}^{m} A_{ni}^k\right)^{\lambda_n/k}. \end{aligned} \tag{3}$$

Since, for $k > 1$ and $a_i > 0$ $(i = 1, \ldots, n)$, we have

$$\frac{\sum_{i=1}^{n} a_i}{\left(\sum_{i=1}^{n} a_i^k\right)^{1/k}} = \sum_{j=1}^{n} \frac{a_j}{\left(\sum_{i=1}^{n} a_i^k\right)^{1/k}} = \sum_{j=1}^{n} \left(\frac{a_j^k}{\sum_{i=1}^{n} a_i^k}\right)^{1/k} \geq \sum_{j=1}^{n} \frac{a_j^k}{\sum_{i=1}^{n} a_i^k} = 1,$$

inequality (2) follows from (3).

From Theorem 3 we obtain immediately Theorem 3 in 2.8.

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2.14.3 Ratios and Differences of Means

Let $a = (a_1, \ldots, a_n)$ and $q = (q_1, \ldots, q_n)$ be two sequences of positive numbers such that $\sum_{i=1}^{n} q_i = 1$. We shall consider the ratio and difference of two weighted means of order s and t $(-\infty < t < s < +\infty)$:

$$Q_{s,t}(a; q) = \frac{M_n^{[s]}(a; q)}{M_n^{[t]}(a; q)}, \tag{1}$$

$$D_{s,t}(a; q) = M_n^{[s]}(a; q) - M_n^{[t]}(a; q). \tag{2}$$

According to Theorem 1 in 2.14.2 we have the following lower bounds

$$1 \leq Q_{s,t}(a; q) \quad \text{and} \quad 0 \leq D_{s,t}(a; q).$$

We shall consider the problem of finding upper bounds for (1) and (2). See 2.11 for the upper bound of (1) when $s = 1$, $t = -1$.

K. Knopp [1] has determined the upper bounds for (1) and (2) in the case $t = 1$, $s > 1$.

The problem of determining upper bounds for (1) for arbitrary s and t, was first solved by W. Specht [2]. He obtained the following result:

Theorem 1. *If $0 < m \leq a_i \leq M$ $(i = 1, \ldots, n)$ and $t < s$, then the following inequality holds*

$$Q_{s,t}(a; q) \leq \Gamma_{s,t}, \tag{3}$$

where $C = M/m$ and

$$\begin{aligned}
\Gamma_{s,t} &= \left(\frac{t(C^s - C^t)}{(s-t)\,(C^t - 1)}\right)^{1/s} \left(\frac{s(C^t - C^s)}{(t-s)\,(C^s - 1)}\right)^{-1/t} && (st \neq 0)\\
&= \left(\frac{C^{s/(C^s-1)}}{e \log\left(C^{s/(C^s-1)}\right)}\right)^{1/s} && (t = 0)\\
&= \left(\frac{C^{t/(C^t-1)}}{e \log\left(C^{t/(C^t-1)}\right)}\right)^{-1/t} && (s = 0).
\end{aligned}$$

G. T. Cargo and O. Shisha [3] have rediscovered this theorem. However, they also considered the equality case in (3), which was not done explicitly in [2]. Their result reads:

Let

$$L = \frac{1}{s-t}\left(\frac{t}{C^t - 1} - \frac{s}{C^s - 1}\right) \quad (st \neq 0)$$
$$= \frac{1}{s \log C} - \frac{1}{C^s - 1} \quad (t = 0)$$
$$= \frac{1}{t \log C} - \frac{1}{C^t - 1} \quad (s = 0).$$

Equality in (3) *for a point* $(a_1, \ldots, a_n)$ *holds if and only if there exists a subsequence* $(k_1, \ldots, k_p)$ *of* $(1, \ldots, n)$ *such that* $\sum_{i=1}^{p} q_{k_i} = L$, $a_{k_i} = M$ $(i = 1, \ldots, p)$ *and* $a_k = m$ *for every* k *distinct from all* k_i.

The upper bound for ratio (1) was also discussed by E. F. BECKENBACH [4].

Finally, let us mention the result due to O. SHISHA and B. MOND [5] which refers to the upper bound of the difference (2).

Theorem 2. *If* $0 < m \leq a_i \leq M$ $(i = 1, \ldots, n)$ *and* $t < s$, *then*

$$D_{s,t}(a; q) \leq \gamma_{s,t}, \tag{4}$$

where

$$\gamma_{s,t} = (\theta M^s + (1-\theta)\, m^s)^{1/s} - (\theta M^t + (1-\theta)\, m^t)^{1/t} \quad (st \neq 0),$$

$$\gamma_{s,0} = (\theta M^s + (1-\theta)\, m^s)^{1/s} - M^\theta m^{1-\theta},$$

$$\gamma_{0,t} = M^\theta m^{1-\theta} - (\theta M^t + (1-\theta)\, m^t)^{1/t}.$$

θ *is defined in the following way. Let*

$$h(x) = x^{1/s} - (ax + b)^{1/t} \quad (st \neq 0)$$
$$= x^{1/s} - m\left(\frac{M}{m}\right)^{(x-m^s)/(M^s-m^s)} \quad (t = 0)$$
$$= -x^{1/} + m\left(\frac{M}{m}\right)^{(x-m^t)/(M^t-m^t)} \quad (s = 0),$$

where

$$a = \frac{M^t - m^t}{M^s - m^s}, \qquad b = \frac{M^s m^t - M^t m^s}{M^s - m^s}.$$

Let J *denote the open interval joining* m^s *to* M^s *if* $s \neq 0$ *and let* $J = (M^t, m^t)$ *if* $s = 0$. *There is an* $x' \in J$ *such that* $h(x) < h(x')$ *for every* $x \in J$ *and* $x \neq x'$. *We set*

$$\theta = \frac{x' - m^s}{M^s - m^s} \quad (s \neq 0) \quad \text{and} \quad \theta = \frac{x' - m^t}{M^t - m^t} \quad (s = 0).$$

Equality in (4) *for a point* $(a_1, \ldots, a_n)$ *holds if and only if there exists a subsequence* $(k_1, \ldots, k_p)$ *of* $(1, \ldots, n)$ *such that* $\sum_{i=1}^{p} q_{k_i} = \theta$, $a_{k_i} = M$ $(i = 1, \ldots, p)$ *and* $a_k = m$ *for every* k *distinct from all* k_i. *Finally, if* $s \geq 1$, *then* x' *is the unique solution of* $h'(x) = 0$ *in* J.

For some interesting generalizations see the paper [6] of E. BECK.

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2.14.4 Refinement of the Arithmetic-Geometric Mean Inequality

First we cite some results which are associated with the inequality

$$A_n(a, p) - G_n(a, p) \geq 0. \tag{1}$$

We shall use the following notations:

$$\Delta_n(a, q) = A_n(a, q) - G_n(a, q),$$

$$S_n(a) = \sum_{1 \leq i < j \leq n} (a_i^{1/2} - a_j^{1/2})^2,$$

$$S_n(a, q) = \sum_{1 \leq i < j \leq n} q_i q_j (a_i^{1/2} - a_j^{1/2})^2,$$

where $a = (a_1, \ldots, a_n)$ is a sequence of nonnegative real numbers and $q = (q_1, \ldots, q_n)$ is a sequence of positive real numbers such that $q_1 + \cdots + q_n = 1$.

H. KOBER [1] has proved the following result:

Theorem 1. *If not all of the numbers* $a_1, \ldots, a_n$ *are equal, the following inequality holds*

$$\frac{\min(q_1, \ldots, q_n)}{n-1} \leq \frac{\Delta_n(a, q)}{S_n(a)} \leq \max(q_1, \ldots, q_n). \tag{2}$$

Proof. Without loss of generality, we can suppose that $q_1 \leq \cdots \leq q_n$. Suppose first that $q_1 < q_2$, and put

$$d_n = \Delta_n(a, q) - \frac{q_1}{n-1} \sum_{1 \leq i < j \leq n} (a_i^{1/2} - a_j^{1/2})^2 = \Delta_n(a, q) - \frac{q_1}{n-1} S_n(a).$$

Then

$$d_n = \sum_{i=2}^{n} (q_i - q_1)\, a_i + \frac{2q_1}{n-1} \sum_{1 \le i < j \le n} (a_i a_j)^{1/2} - G_n(a, q)\,.$$

Since

$$\sum_{i=2}^{n} (q_i - q_1) + \frac{n(n-1)}{2} \cdot \frac{2q_1}{n-1} = 1,$$

applying the inequality between the weighted arithmetic and geometric means for the case when the sum of the weights is 1, we get $d_n \ge 0$, with equality if and only if

$$a_2 = \cdots = a_n = (a_1 a_2)^{1/2} = \cdots = (a_{n-1} a_n)^{1/2},$$

i.e., if and only if $a_2 = \cdots = a_n = 0$ and $a_1 > 0$.

Similarly we proceed in the case $q_1 = q_2 < q_3$, etc.

In this way the first inequality of (2) is proved.

Let us now prove the second inequality of (2). This proof is based on the fact that, for $x_i \ge 0$,

$$(3) \qquad (n-2) \sum_{i=1}^{n} x_i + n(x_1 \cdots x_n)^{1/n} - 2 \sum_{1 \le i < j \le n} (x_i x_j)^{1/2} \ge 0,$$

with equality if and only if $x_i = 0$ and $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n$. For the proof of this inequality see 3.9.69.

Since

$$S_n(a) = nA_n(a) - nG_n(a) + \left\{ (n-2) \sum_{i=1}^{n} a_i + n(a_1 \cdots a_n)^{1/n} - 2 \sum_{1 \le i < j \le n} (a_i a_j)^{1/2} \right\},$$

applying inequality (3) we see that the expression in curled brackets is always positive except in the case when one of the numbers $a_1, \ldots, a_n$ is equal to zero, and all the others are equal. Therefore

$$\frac{\Delta_n(a, q)}{S_n(a)} < \frac{\Delta_n(a, q)}{nD_n(a)},$$

where $D_n(a) = A_n(a) - G_n(a)$.

We have two cases:

1° $q_1 = \cdots = q_n = 1/n$; then $\Delta_n(a, q) = D_n(a)$ and the second inequality of (2) is also proved.

2° Suppose that not all q_i are equal. Since

$$D_n(a) - \frac{\Delta_n(a, q)}{nq_n} = \frac{1}{nq_n} \sum_{i=1}^{n-1} (q_n - q_i)\, a_i + \frac{1}{nq_n} \prod_{i=1}^{n} a_i^{q_i} - \prod_{i=1}^{n} a_i^{1/n},$$

applying the inequality between the arithmetic and geometric means we get $D_n(a) - \frac{\Delta_n(a, q)}{nq_n} \geq 0$ and in this case the second inequality of (2) is proved.

Finally, if $q_2 = \cdots = q_n > q_1$, then

$$D_n(a) - \frac{\Delta_n(a, q)}{nq_n} = 0$$

if and only if $x_1 = 0$, $x_2 = \cdots = x_n$, which together with inequality (3) completes the proof.

H. Kober has proved that the upper bound in (2) is always reached.

P. H. Diananda [2] has, however, proved that this is not always the case with the lower bound.

P. H. Diananda has proved the following result:

Theorem 2. *If not all the members of the sequence a are equal, the following inequalities hold:*

$$\frac{1}{1 - \min(q_1, \ldots, q_n)} \leq \frac{\Delta_n(a, q)}{S_n(a, q)} \leq \frac{1}{\min(q_1, \ldots, q_n)}. \tag{4}$$

Both bounds in (4) *are always reached.*

Let us finally mention a result proved by P. H. Diananda [3] which is a generalization of Theorem 2, namely,

Theorem 3. *The following inequalities hold*

$$\frac{\min\left(\frac{q_1}{q_n'}, \ldots, \frac{q_n}{q_n'}\right)}{1 - \min(q_1', \ldots, q_n')} \leq \frac{\Delta_n(a, q)}{S_n(a, q')} \leq \frac{\max\left(\frac{q_1}{q_1'}, \ldots, \frac{q_n}{q_n'}\right)}{\min(q_1', \ldots, q_n')}, \tag{5}$$

where $q' = (q_1', \ldots, q_n')$ *is a sequence of positive numbers such that* $q_1' + \cdots + q_n' = 1$.

Both bounds in (5) *are always reached.*

For other results which refer to inequalities of the type

$$\Delta_n(a; q) \geq \alpha \sum_{1 \leq i < j \leq n} \beta_{ij} (a_i^{1/2} - a_j^{1/2}),$$

where α and β_{ij} are some rational functions of $q_1, \ldots, q_n$, see article [4] of A. Dinghas.

We shall now quote, without proof, some results which refine the inequality

$$\frac{A_n(a)}{G_n(a)} \geq 1. \tag{6}$$

Let $a = (a_1, \ldots, a_n)$ be a sequence of positive numbers. We shall use the following notation:

$$\sigma = \sum_{1 \le i < j \le n} (a_i - a_j)^2,$$

$$\Delta = \prod_{1 \le i < j \le n} (a_i - a_j)^2,$$

$$P_n(t) = \frac{1}{n!} \prod_{k=0}^{n-2} \left(\frac{t+k}{n-k}\right)^{n-k-1},$$

$$Q_n(t) = \prod_{k=1}^{n} \left(1 + \frac{n-k}{t+k-1}\right).$$

C. L. SIEGEL [5] has proved the following result:

Theorem 4. *If $n \ge 2$ and $\Delta \neq 0$, then*

$$A_n(a)^n > G_n(a)^n Q_n(\mu), \tag{7}$$

where μ is the only positive root of the equation

$$P_n(\mu)\,\Delta = G_n(a)^{n(n-1)}. \tag{8}$$

Since $Q_n(t) > 1$ for all positive t, inequality (7) is really a refinement of inequality (6).

Inequality (7) is at the same time a refinement of an inequality of I. SCHUR [6]:

$$A_n(a)^{n(n-1)} > \Delta R_n(0), \tag{9}$$

where

$$R_n(t) = \prod_{k=1}^{n} \frac{(t+n-1)^{n-1}}{k^k (t+k-1)^{k-1}}.$$

Indeed, by eliminating $G_n(a)$ from (7) and (8) we get

$$A_n(a)^{n(n-1)} \ge \Delta R_n(\mu),$$

which is sharper than (9) since $R_n(t)$ is an increasing function with respect to t.

The following result due to J. HUNTER [7] is in a way related to Theorem 4:

Theorem 5. *Let μ be the root of the equation*

$$\sigma = \mu^2 (n-1) \left(nA_n(a)\right)^2,$$

which belongs to the interval $(0, 1)$. *Then, we have*

$$\left(\frac{A_n(a)}{G_n(a)}\right)^n \ge \frac{1}{(1+\mu(n-1))(1-\mu)^{n-1}}.$$

The results obtained by A. DINGHAS [4] follow from above. In this paper inequalities involving elementary symmetric functions were also considered.

We notice that all the known proofs of the above results are rather complicated and it would therefore be of interest to find some simpler proofs.

Remark. Refinements of (6) analogous to Theorems 1 and 2 have been given by P. S. BULLEN in Pacific J. Math. 15, 47—54 (1965).

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2.14.5 Some General Inequalities Involving Means

Let I and J denote finite nonempty sets of positive integers, let $N(I)$ be the number of elements of I, let $\emptyset$ denote the empty set, and let $I_p = \{1, \ldots, p\}$ $(p \geq 1)$. Furthermore, let $a = (a_1, \ldots, a_n)$ $(n \geq 1)$ be a sequence of real non-negative numbers.

We shall consider the mean of order r, which is formed from a subsequence of a, for the case $0 \leq r < +\infty$. Then

$$M_I^{[r]}(a) = N(I)^{-1/r}\Big(\sum_{i\in I} a_i^r\Big)^{1/r} \quad (0 < r < +\infty), \tag{1}$$

$$M_I^{[0]}(a) = \Big(\prod_{i\in I} a_i\Big)^{1/N(I)}.$$

We shall also consider the difference

$$\varrho_{r,s}(a; I) = N(I)\left(M_I^{[r]}(a) - M_I^{[s]}(a)\right) \tag{2}$$

for $0 \leq s \leq r < +\infty$.

Since $M_I^{[r]}(a)$ is an increasing function of r, we have

$$\varrho_{r,s}(a; I) \geq 0,$$

equality occuring only in one of the following cases:

1° if $r = s$,

2° if all a_n $(n \in I)$ are equal.

W. N. Everitt [1] has proved the following three theorems:

Theorem 1. *Let $I \cap J = \emptyset$. Then:*

1° *for* $1 < r < +\infty$,

$$N(I \cup J)\, M_{I \cup J}^{[r]}(a) \geq N(I)\, M_I^{[r]}(a) + N(J)\, M_J^{[r]}(a). \tag{3}$$

Equality holds if and only if

$$M_I^{[r]}(a) = M_J^{[r]}(a); \tag{4}$$

2° *for* $r = 1$ *and* $I \cap J = \emptyset$, *equality always holds in* (3);

3° *for* $0 \leq r < 1$, *inequality in* (3) *reverses, equality holding if and only if condition* (4) *is fulfilled.*

Theorem 2. *If* $0 \leq s \leq 1 \leq r < +\infty$, *then*

$$\varrho_{r,s}(a; I \cup J) \geq \varrho_{r,s}(a; I) + \varrho_{r,s}(a; J). \tag{5}$$

Equality holds if and only if one of the following conditions is fulfilled:

1° $s = r = 1$ *and* $I \cap J = \emptyset$;

2° $0 \leq s < 1 < r < +\infty$ *and*

$$M_I^{[r]}(a) = M_J^{[r]}(a), \tag{6}$$

$$M_I^{[s]}(a) = M_J^{[s]}(a); \tag{7}$$

3° $0 \leq s < 1 = r$ *and* (7);

4° $s = 1 < r < +\infty$ *and* (6).

Theorem 3. *If* $0 \leq s \leq 1 \leq r < +\infty$, *for all* $p \geq 1$, *we have*

$$\varrho_{r,s}(a; I_{p+1}) \geq \varrho_{r,s}(a; I_p),$$

with equality if and only if one of the following four conditions is fulfilled:

1° $s = r = 1$;

2° $0 \leq s < 1 < r < +\infty$ *and all the numbers* a_n $(1 \leq n \leq p + 1)$ *are equal;*

3° $0 \leq s < 1 = r, \quad a_{p+1} = M_{I_p}^{[r]}(a)$;

4° $s = 1 < r < +\infty, a_{p+1} = M_{I_p}^{[s]}(a)$.

Unifying the results proved by W. N. Everitt [1] and D. S. Mitrinović and P. M. Vasić [2], H. W. McLaughlin and F. T. Metcalf [3] obtained some interesting inequalities for means of order r. Later, D. S. Mitrinović and P.M. Vasić [4] proved even more general results which contain inequalities of H. W. McLaughlin and F. T. Metcalf from [3] and some inequalities of P. S. Bullen [5]. We give the results of Mitrinović and Vasić:

Let I be a nonempty subset of the set of natural numbers. The weighted mean of order r (r is real and finite) is defined by

$$M_I^{[r]}(a;p) = \left(\frac{\sum_I p_i a_i^r}{\sum_I p_i}\right)^{1/r} \quad (r \neq 0), \qquad M_I^{[0]}(a;p) = \left(\prod_I a_i^{p_i}\right)^{1/\sum_I p_i},$$

where $a_i > 0$, $p_i > 0$ $(i \in I)$ and $\sum_I = \sum_{i \in I}$. We also write $M_I^{[r]}(a;p)^{\lambda r}, \ldots$ for $(M_I^{[r]}(a;p))^{\lambda r}, \ldots$

Theorem 4. *If $\lambda + \mu \geq 1$ with $\lambda > 0$, $\mu > 0$, and if I_1, I_2, J_1, J_2 are subsets of N such that $I_1 \cap J_1 = \emptyset$ and $I_2 \cap J_2 = \emptyset$, then*

$$\begin{aligned}(8)\qquad &\Big(\sum_{I_1 \cup J_1} p_i\Big)^\lambda \Big(\sum_{I_2 \cup J_2} q_i\Big)^\mu M_{I_1 \cup J_1}^{[r]}(a;p)^{\lambda r} M_{I_2 \cup J_2}^{[s]}(b;q)^{\mu s} \\ &\geq \Big(\sum_{I_1} p_i\Big)^\lambda \Big(\sum_{I_2} q_i\Big)^\mu M_{I_1}^{[r]}(a;p)^{\lambda r} M_{I_2}^{[s]}(b;q)^{\mu s} \\ &+ \Big(\sum_{J_1} p_i\Big)^\lambda \Big(\sum_{J_2} q_i\Big)^\mu M_{J_1}^{[r]}(a;p)^{\lambda r} M_{J_2}^{[s]}(b;q)^{\mu s}.\end{aligned}$$

Equality takes place if and only if $\lambda + \mu = 1$, $\lambda, \mu > 0$ and

$$\frac{\left(\sum_{I_1} p_i\right) M_{I_1}^{[r]}(a;p)^r}{\left(\sum_{I_2} q_i\right) M_{I_2}^{[s]}(b;q)^s} = \frac{\left(\sum_{J_1} p_i\right) M_{J_1}^{[r]}(a;p)^r}{\left(\sum_{J_2} q_i\right) M_{J_2}^{[s]}(b;q)^s}.$$

If $\lambda + \mu = 1$ and $\lambda > 1$, or $\lambda + \mu = 1$ and $\lambda < 0$, the opposite inequality holds.

Proof. Putting $n = 2$, $m = 2$ and

$$A_{11} = \Big(\sum_{I_1} p_i\Big) M_{I_1}^{[r]}(a;p)^r, \quad A_{12} = \Big(\sum_{J_1} p_i\Big) M_{J_1}^{[r]}(a;p)^r,$$
$$A_{21} = \Big(\sum_{I_2} q_i\Big) M_{I_2}^{[s]}(b;q)^s, \quad A_{22} = \Big(\sum_{J_2} q_i\Big) M_{J_2}^{[s]}(b;q)^s,$$

then since

$$\Big(\sum_I p_i\Big) M_I^{[r]}(a;p)^r + \Big(\sum_J p_i\Big) M_J^{[r]}(a;p)^r = \Big(\sum_{I \cup J} p_i\Big) M_{I \cup J}^{[r]}(a;p)^r,$$

from (2) in 2.14.2 follows (8).

Corollary 1. *If $I_1 = I_2 = I$, $J_1 = J_2 = J$ and $I \cap J = \emptyset$, inequality* (8) *becomes*

$$f(I \cup J) \geq f(I) + f(J),$$

where

$$f(I) = \Big(\sum_I p_i\Big)^\lambda \Big(\sum_I q_i\Big)^\mu M_I^{[r]}(a;p)^{\lambda r} M_I^{[s]}(b;q)^{\mu s}.$$

The function f is, therefore, superadditive.

Corollary 2. *If $I_1 = J_2 = I$, $J_1 = I_2 = J$ and $I \cap J = \emptyset$, inequality* (8) *becomes*

$$\Big(\sum_{I\cup J} p_i\Big)^\lambda \Big(\sum_{I\cup J} q_i\Big)^\mu M_{I\cup J}^{[r]}(a;p)^{\lambda r} M_{I\cup J}^{[s]}(b;q)^{\mu s}$$
$$\geq \Big(\sum_I p_i\Big)^\lambda \Big(\sum_J q_i\Big)^\mu M_I^{[r]}(a;p)^{\lambda r} M_J^{[s]}(b;q)^{\mu s}$$
$$+ \Big(\sum_J p_i\Big)^\lambda \Big(\sum_I q_i\Big)^\mu M_J^{[r]}(a;p)^{\lambda r} M_I^{[s]}(b;q)^{\mu s}.$$

This inequality states the following property

$$f(I, J) + f(J, I) \leq f(I \cup J, J \cup I),$$

where

$$f(I, J) = \Big(\sum_I p_i\Big)^\lambda \Big(\sum_J q_i\Big)^\mu M_I^{[r]}(a, p)^{\lambda r} M_J^{[s]}(b, q)^{\mu s}.$$

Theorem 5. *If $\lambda + \mu \geq 1, \lambda > 0, \mu > 0$ and if $I_{11}, \ldots, I_{m1}$* and *$I_{12}, \ldots, I_{m2}$ are nonempty subsets of the set of natural numbers N, such that for $k, j = 1, \ldots, m$ with $j \neq k$,*

$$I_{k1} \cap I_{j1} = \emptyset, \quad I_{k2} \cap I_{j2} = \emptyset,$$

then

$$\Big(\sum_{I_{11}\cup\cdots\cup I_{m1}} p_i\Big)^\lambda \Big(\sum_{I_{12}\cup\cdots\cup I_{m2}} q_i\Big)^\mu M_{I_{11}\cup\cdots\cup I_{m1}}^{[r]}(a;p)^{\lambda r} M_{I_{12}\cup\cdots\cup I_{m2}}^{[s]}(b;q)^{\mu s}$$
$$\geq \Big(\sum_{I_{11}} p_i\Big)^\lambda \Big(\sum_{I_{12}} q_i\Big)^\mu M_{I_{11}}^{[r]}(a;p)^{\lambda r} M_{I_{12}}^{[s]}(b;q)^{\mu s}$$
$$+ \cdots$$
$$+ \Big(\sum_{I_{m1}} p_i\Big)^\lambda \Big(\sum_{I_{m2}} q_i\Big)^\mu M_{I_{m1}}^{[r]}(a;p)^{\lambda r} M_{I_{m2}}^{[s]}(b;q)^{\mu s}.$$

Equality holds if and only if $\lambda + \mu = 1$ ($\lambda > 0$ and $\mu > 0$), and if the sequences

$$\Big(\sum_{I_{i1}} p_i\Big) M_{I_{i1}}^{[r]}(a;p)^r \quad (i = 1, \ldots, m),$$
$$\Big(\sum_{I_{i2}} q_i\Big) M_{I_{i2}}^{[s]}(b;q)^s \quad (i = 1, \ldots, m)$$

are proportional.

The proof of Theorem 5 is analogous to the proof of Theorem 4, and is obtained from inequality (2) in 2.14.2 with $n = 2$.

Theorem 6. *If $\lambda_1 + \cdots + \lambda_n \geq 1$, $\lambda_i > 0$ for $i = 1, \ldots, n$ and if I_{jk} are nonempty subsets of the set of natural numbers N, such that*

$$I_{k1} \cap I_{j1} = \emptyset, \ldots, I_{kn} \cap I_{jn} = \emptyset$$

for $k, j = 1, \ldots, m$ *and* $k \neq j$, *then*

$$(9)\qquad \Big(\sum_{I_{11}\cup\cdots\cup I_{m1}} p_{1i}\Big)^{\lambda_1} \cdots \Big(\sum_{I_{1n}\cup\cdots\cup I_{mn}} p_{mi}\Big)^{\lambda_n} M^{[r_1]}_{I_{11}\cup\cdots\cup I_{m1}}(a_1; p_1)^{\lambda_1 r_1} \cdots M^{[r_n]}_{I_{1n}\cup\cdots\cup I_{mn}}(a_n; p_n)^{\lambda_n r_n}$$
$$\geq \Big(\sum_{I_{11}} p_{1i}\Big)^{\lambda_1} \cdots \Big(\sum_{I_{1n}} p_{mi}\Big)^{\lambda_n} M^{[r_1]}_{I_{11}}(a_1; p_1)^{\lambda_1 r_1} \cdots M^{[r_n]}_{I_{1n}}(a_n; p_n)^{\lambda_n r_n}$$
$$+ \cdots$$
$$+ \Big(\sum_{I_{m1}} p_{1i}\Big)^{\lambda_1} \cdots \Big(\sum_{I_{mn}} p_{mi}\Big)^{\lambda_n} M^{[r_1]}_{I_{m1}}(a_1; p_1)^{\lambda_1 r_1} \cdots M^{[r_n]}_{I_{mn}}(a_n; p_n)^{\lambda_n r_n}.$$

Proof. Inequality (9) can be obtained by putting in (2) in 2.14.2

$$A_{kj} = \Big(\sum_{I_{jk}} p_{ki}\Big) M^{[r_k]}_{I_{jk}}(a_k; p_k)^{r_k} \qquad (j = 1, \ldots, m;\ k = 1, \ldots, n).$$

Theorem 5 is, of course, merely a special case of Theorem 6 with $n = 2$.

As has been said before, Theorems 4, 5 and 6 contain a number of known results. We shall mention the following particular results which are included in those theorems (see D. S. Mitrinović and P. M. Vasić [2]).

Theorem 7. *If* $rs < 0$, *then*

$$(10)\qquad \frac{\Big(\sum\limits_{\nu=1}^{n} p_\nu\Big)^{\frac{s}{s-r}}}{\Big(\sum\limits_{\nu=1}^{n} q_\nu\Big)^{\frac{r}{s-r}}} \left(\frac{M_n^{[r]}(a;p)}{M_n^{[s]}(a;q)}\right)^{\frac{rs}{s-r}} \geq \frac{\Big(\sum\limits_{\nu=1}^{n-1} p_\nu\Big)^{\frac{s}{s-r}}}{\Big(\sum\limits_{\nu=1}^{n-1} q_\nu\Big)^{\frac{r}{s-r}}} \left(\frac{M_{n-1}^{[r]}(a;p)}{M_{n-1}^{[s]}(a;q)}\right)^{\frac{rs}{s-r}} + \frac{p_n^{\frac{s}{s-r}}}{q_n^{\frac{r}{s-r}}}.$$

If $rs > 0$, *the inequality is reversed.*
Equality holds if and only if

$$(11)\qquad a_n \Big(\frac{p_n}{q_n}\Big)^{\frac{1}{r-s}} = a_{n-1}\Big(\frac{p_{n-1}}{q_{n-1}}\Big)^{\frac{1}{r-s}} = \cdots = a_1\Big(\frac{p_1}{q_1}\Big)^{\frac{1}{r-s}}.$$

Theorem 8. *For* $r > 0$,

$$(12)\qquad \left(\frac{\sum\limits_{\nu=1}^{n} p_\nu}{\sum\limits_{\nu=1}^{n} q_\nu}\right)^{\frac{1}{r}\left(\sum\limits_{\nu=1}^{n} q_\nu\right)} \times \left(\frac{M_n^{[r]}(a;p)}{G_n(a;q)}\right)^{\sum\limits_{\nu=1}^{n} q_\nu}$$
$$\geq \Big(\frac{p_n}{q_n}\Big)^{\frac{q_n}{r}} \left(\frac{\sum\limits_{\nu=1}^{n-1} p_\nu}{\sum\limits_{\nu=1}^{n-1} q_\nu}\right)^{\frac{1}{r}\left(\sum\limits_{\nu=1}^{n-1} q_\nu\right)} \times \left(\frac{M_{n-1}^{[r]}(a;p)}{G_{n-1}(a;q)}\right)^{\sum\limits_{\nu=1}^{n-1} q_\nu};$$

with equality if and only if equalities* (11) *hold.

Notice that inequalities (10) and (12) can be iterated, so that we get, for example,

$$\frac{M_n^{[r]}(a;p)}{M_n^{[s]}(a;q)} \geq \left(\frac{\left(\sum_{\nu=1}^{n} q_\nu\right)^{\frac{r}{s-r}}}{\left(\sum_{\nu=1}^{n} p_\nu\right)^{\frac{s}{s-r}}} \sum_{\nu=1}^{n} \frac{p_\nu^{\frac{s}{s-r}}}{q_\nu^{\frac{r}{s-r}}}\right)^{\frac{s-r}{rs}};$$

and Theorem 2 in 2.14.2 follows from this inequality if we put $p_i = q_i$ $(i = 1, \ldots, n)$.

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2.14.6 The λ-Method of Mitrinović and Vasić

This method (see [1]) can be summarized as follows:

1° Start with an inequality which can be proved by the theory of maxima and minima;

2° In a convenient manner introduce one or more parameters into the function from which that inequality was obtained;

3° Find the extreme values of such a function, treating the parameters as fixed.

In this way an inequality involving one or more parameters is obtained. Assigning conveniently chosen values to those parameters, one may obtain various inequalities whose forms bear no similarity to the original. This method often unifies isolated inequalities and yields known inequalities as special cases.

We shall demonstrate the method by a simple example.

The inequality

$$nA_n(a) - nM_n^{[r]}(a) \geq (n-1)\, A_{n-1}(a) - (n-1)\, M_{n-1}^{[r]}(a),$$

which holds for $r < 1$ (the inequality is reversed for $r > 1$) can be proved by the method of examining functions, starting with

$$g(a_n) = nA_n(a) - nM_n^{[r]}(a). \tag{1}$$

Instead of g, we shall consider the function

$$f(a_n) = nA_n(a) - \lambda n M_n^{[r]}(a), \tag{2}$$

where $\lambda > 0$ is a parameter.

From (2) we get

$$f'(a_n) = 1 - \lambda a_n^{r-1} \left(M_n^{[r]}(a)\right)^{1-r},$$

$$f''(a_n) = -\lambda(r-1)\frac{n-1}{n} a_n^{r-2} \left(M_n^{[r]}(a)\right)^{1-2r} \left(M_{n-1}^{[r]}(a)\right)^r.$$

If $0 < \lambda^{\frac{r}{1-r}} < n$, the only positive zero of the function f' is

$$a_n = \lambda^{\frac{1}{1-r}} M_{n-1}^{[r]}(a) \left(\frac{n-1}{n - \lambda^{\frac{r}{1-r}}}\right)^{1/r}.$$

Since for $r > 1$ and $\lambda > 0$, $f''(a_n) < 0$, we get the following inequality

$$\begin{aligned} &nA_n(a) - \lambda n M_n^{[r]}(a) \\ &\le (n-1) A_{n-1}(a) - \lambda(n-1)\left((n-1)^{\frac{1-r}{r}}\left(n - \lambda^{\frac{r}{1-r}}\right)^{\frac{r-1}{r}}\right) M_{n-1}^{[r]}(a), \end{aligned} \tag{3}$$

where $r > 1$ and $0 < \lambda^{\frac{r}{1-r}} < n$.

If $r < 1$, the sign $\le$ in (3) should be replaced by $\ge$. In this case, excluding the case $r = 0$, the inequality is obtained from previous considerations. For $r = 0$, the following inequality holds

$$nA_n(a) - \lambda n G_n(a) \ge (n-1) A_{n-1}(a) - \lambda^{\frac{n}{n-1}} (n-1) G_{n-1}(a) \quad (\lambda > 0), \tag{4}$$

which can be obtained from (3) by letting $r \to 0$, or by examining the corresponding functions, as in the proof of (3).

Putting $\lambda = A_n(a)/M_n^{[r]}(a)$ in (3), we get

$$n\left(\frac{M_n^{[r]}(a)}{A_n(a)}\right)^{\frac{r}{1-r}} \ge (n-1)\left(\frac{M_{n-1}^{[r]}(a)}{A_{n-1}(a)}\right)^{\frac{r}{1-r}} + 1 \qquad (r < 0).$$

From this inequality, using a_i^k instead of a_i and setting $s = rk$, we get the following, more general, inequality

$$n\left(\frac{M_n^{[s]}(a)}{M_n^{[k]}(a)}\right)^{\frac{ks}{k-s}} \ge (n-1)\left(\frac{M_{n-1}^{[s]}(a)}{M_{n-1}^{[k]}(a)}\right)^{\frac{ks}{k-s}} + 1 \qquad (ks < 0).$$

This inequality has been proved in [2].

Putting $\lambda = A_n(a)/G_n(a)$ in (4), we get

$$\left(\frac{G_n(a)}{A_n(a)}\right)^n \leq \left(\frac{G_{n-1}(a)}{A_{n-1}(a)}\right)^{n-1},$$

which has been proved many times (see, for example, [3] and [4]).

By a similar method, and by a suitable choice of the parameter λ, other classical and new inequalities can be obtained from proven inequalities.

Up to now, for the sake of simplicity, we have considered only the most elementary cases. We shall now quote some more general results, without proof, using the following notation:

$$P_n = \sum_{i=1}^{n} p_i \quad \text{and} \quad Q_n = \sum_{i=1}^{n} q_i.$$

Theorem 1 (D. S. Mitrinović and P. M. Vasić [1]). *If $r < 1$, the following inequality holds:*

$$Q_n A_n(a;q) - \lambda \frac{q_n}{p_n} P_n M_n^{[r]}(a;p)$$

$$\geq Q_{n-1}A_{n-1}(a;q) - \lambda \frac{q_n}{p_n} P_{n-1} \left\{ P_{n-1}^{\frac{1-r}{r}} \left(P_n - p_n \lambda^{\frac{r}{1-r}} \right)^{\frac{r-1}{r}} \right\} M_{n-1}^{[r]}(a;p),$$

where $0 < \lambda^{\frac{r}{1-r}} < \frac{1}{p_n} P_n$.

For $r > 1$, the sign of the inequality is reversed.

Theorem 2 (D. S. Mitrinović and P. M. Vasić [1], P. S. Bullen [5]). *For any $\lambda > 0$ we have*

$$Q_n A_n(a;q) - \lambda \frac{q_n}{p_n} P_n G_n(a;p)$$

$$\geq Q_{n-1}A_{n-1}(a;q) - \lambda^{P_n/P_{n-1}} \frac{q_n}{p_n} P_{n-1} G_{n-1}(a;p).$$

Theorem 3 (P. S. Bullen [5]). *For any $\lambda > 0$ we have*

$$\frac{(A_n(a;q) + \lambda)^{Q_n}}{(G_n(a;p))^{q_n P_n/p_n}} \geq \frac{(A_{n-1}(a;q) + \lambda Q_n/Q_{n-1})^{Q_{n-1}}}{(G_{n-1}(a;p))^{q_n P_n/p_n}},$$

with equality only when $A_n(a;q) = a_n + \lambda$.

Theorem 4 (D. S. Mitrinović and P. M. Vasić [6]). *Let λ and μ be two real numbers such that $\lambda\mu > 0$. If $\mu p_n - P_n < 0$, we have*

$$Q_n A_n(a; q) - \lambda \frac{q_n}{p_n} P_n \big(G_n(a; p)\big)^{\mu} \geq Q_{n-1} A_{n-1}(a; q)$$
$$- (\lambda\mu)^{\frac{P_n}{P_n - \mu p_n}} \frac{q_n}{p_n} \big(G_{n-1}(a; p)\big)^{\frac{\mu P_{n-1}}{P_n - \mu p_n}} \left(\frac{P_n}{\mu} - p_n\right).$$

In the case where $\mu p_n - P_n > 0$, the inequality is reversed.
In both cases equality holds if and only if

$$a_n = (\lambda\mu)^{\frac{P_n}{P_n - \mu p_n}} \big(G_{n-1}(a; p)\big)^{\frac{\mu P_{n-1}}{P_n - \mu p_n}}.$$

Theorem 5 (D. S. Mitrinović and P. M. Vasić [6]). *Let α, β, λ be real numbers such that $\lambda > 0$ and $\beta(\alpha - \beta p_n) > 0$. If $\alpha - \beta p_n > 0$, we have*

$$\frac{(A_n(a; q) + \lambda)^{\alpha}}{(G_n(a; p))^{\beta P_n}}$$
$$\geq \left(\frac{q_n}{\beta p_n}\right)^{\beta p_n} \left(\frac{\alpha}{Q_n}\right)^{\alpha} \left(\frac{Q_{n-1}}{\alpha - \beta p_n}\right)^{\alpha - \beta p_n} \frac{(A_{n-1}(a; q) + \lambda Q_n / Q_{n-1})^{\alpha - \beta p_n}}{(G_{n-1}(a; p))^{\beta P_{n-1}}}.$$

In the case when $\alpha - \beta p_n < 0$, the inequality is reversed.
In both cases equality holds if and only if

$$(\alpha - \beta p_n)\, q_n a_n = \beta p_n (Q_{n-1} A_{n-1}(a; q) + \lambda Q_n).$$

Theorem 6 (P. S. Bullen [7]). *If $r < s$, $s \neq 0, +\infty$ and $0 < p_n \lambda^{\frac{r}{s-r}} < P_n$, then*

$$Q_n \big(M_n^{[s]}(a; q)\big)^s - \lambda \frac{q_n}{p_n} P_n \big(M_n^{[r]}(a; p)\big)^s \geq Q_{n-1} \big(M_{n-1}^{[s]}(a; q)\big)^s$$
$$- \lambda \frac{q_n}{p_n} P_{n-1} \big(M_{n-1}^{[r]}(a; p)\big)^s P_{n-1}^{\frac{s-r}{r}} \left(P_n - p_n \lambda^{\frac{r}{s-r}}\right)^{\frac{r-s}{r}},$$

with equality only when $\sum_{\nu=1}^{n-1} a_\nu^r p_\nu = a_n^r \left(\lambda^{\frac{r}{r-s}} P_n - p_n\right)$.

If $r > s$ then the inequality is reversed.

In the following theorem we use the notations:

$$a = (a_1, \ldots, a_{n+m}), \qquad \bar{a} = (a_1, \ldots, a_n), \qquad \tilde{a} = (a_{n+1}, \ldots, a_{n+m}),$$

$$P_{n+m} = p_1 + \cdots + p_{n+m},\ \bar{P}_n = p_1 + \cdots + p_n,\ \tilde{P}_m = p_{n+1} + \cdots + p_{n+m},$$

$$Q_{n+m} = q_1 + \cdots + q_{n+m},\ \bar{Q}_n = q_1 + \cdots + q_n,\ \tilde{Q}_m = q_{n+1} + \cdots + q_{n+m}.$$

Theorem 7 (P. S. Bullen [7]). *If* $\frac{s}{r} > 1$, *and* $0 < \tilde{P}_m \lambda^{\frac{r}{s-r}} < P_{n+m}$, *then*

$$\frac{Q_{n+m}}{\tilde{Q}_m} (M_{n+m}^{[s]}(a;q))^s - \frac{P_{n+m}}{\tilde{P}_m} (M_{n+m}^{[r]}(a;p))^s$$

$$\geq \frac{\bar{Q}_n}{\tilde{Q}_m} (M_n^{[s]}(\bar{a};\bar{q}))^s - \frac{\bar{P}_n}{\tilde{P}_m} (M_n^{[r]}(\bar{a};\bar{p}))^s \bar{P}_n^{\frac{s-r}{r}} \left(P_{n+m} - \tilde{P}_m \lambda^{\frac{r}{s-r}}\right)^{\frac{r-s}{r}}$$

$$+ (M_m^{[s]}(\tilde{a};\tilde{q}))^s - \lambda^{-1} (M_m^{[r]}(\tilde{a};\tilde{p}))^s,$$

$$\frac{Q_{n+m}}{\tilde{Q}_m} (M_{n+m}^{[s]}(a;q))^s - \lambda \frac{P_{n+m}}{\tilde{P}_m} (M_{n+m}^{[r]}(a;p))^s \frac{(M_m^{[s]}(\tilde{a};\tilde{q}))^s}{(M_m^{[r]}(\tilde{a};\tilde{p}))^s}$$

$$\geq \frac{\bar{Q}_n}{\tilde{Q}_m} (M_n^{[s]}(\bar{a};\bar{q}))^s - \lambda \frac{\bar{P}_n}{\tilde{P}_m} (M_n^{[r]}(\bar{a};\bar{p}))^s$$

$$\times \frac{(M_m^{[s]}(\tilde{a};\tilde{q}))^s}{(M_m^{[r]}(\tilde{a};\tilde{p}))^s} \bar{P}_n^{\frac{s-r}{r}} \left(P_{n+m} - \tilde{P}_m \lambda^{\frac{r}{s-r}}\right)^{\frac{r-s}{r}}.$$

Remark 1. The above Theorem 7 which appeared in [7] is given here in the corrected form, at the request of P. S. Bullen.

Remark 2. Inequalities whose form is analogous to

$$nA_n(a) - nG_n(a) \geq (n-1) A_{n-1}(a) - (n-1) G_{n-1}(a)$$

are sometimes called Rado type inequalities, while inequalities analogous to

$$\left(\frac{G_n(a)}{A_n(a)}\right)^n \leq \left(\frac{G_{n-1}(a)}{A_{n-1}(a)}\right)^{n-1}$$

are sometimes called Popoviciu type inequalities.

Those names are often used in 2.15.2.

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2.15 Symmetric Means and Functions

2.15.1 Definitions and Main Relations between Symmetric Means

Definition 1. *Let $a = (a_1, \ldots, a_n)$ be a positive sequence. Then the r-th symmetric function is defined by*

$$e_r = E_n^{[r]} = e_r(a) = E_n^{[r]}(a) = \sum \prod_{j=1}^{r} a_{i_j}, \tag{1}$$

the sum being over all r-tuples $i_1, \ldots, i_r$, such that $1 \le i_1 < \cdots < i_r \le n$; in addition we define $e_0 = 1$. The r-th symmetric mean is

$$p_r = P_n^{[r]} = \frac{E_n^{[r]}}{\binom{n}{r}} \quad \textit{for} \quad r = 0, 1, \ldots, n. \tag{2}$$

A series of simple but remarkable inequalities follows from the following simple observation; see [1], p. 11, [2], p. 104, and [3], pp. 115–117.

Theorem 1. *If $f(x) = \sum_{i=0}^{n} c_i x^i$ has n real roots and if $c_i = \binom{n}{i} d_i$, for $i = 1, \ldots, n-1$, then*

$$d_i^2 - d_{i-1}\, d_{i+1} \ge 0, \tag{3}$$

the inequality being strict unless all the roots are identical.

Inequality (3) implies the weaker result (see [2], p. 52, and [3], p. 117)

$$c_i^2 - c_{i-1}c_{i+1} > 0 \quad \text{for} \quad i = 1, \ldots, n-1. \tag{4}$$

Theorem 2. *For $r = 1, \ldots, n-1$ we have*

$$p_r^2 - p_{r-1}p_{r+1} \ge 0, \tag{5}$$

and

$$e_r^2 - e_{r-1}e_{r+1} > 0, \tag{6}$$

with equality in (5) *if and only if $a_1 = \cdots = a_n$.*

Proof. Since

$$\sum_{k=0}^{n} e_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} p_k x^{n-k} = \prod_{k=1}^{n} (a_k + x),$$

this result is an immediate corollary of Theorem 1.

Corollary 1. *If* $r < s$, *then* $e_{r-1}e_s < e_r e_{s-1}$.

Corollary 2. *If* $1 \leq r \leq n-1$ *and* $e_{r-1} > e_r$, *then* $e_r > e_{r+1}$.

Proof. These are immediate corollaries of (6).

If b is any sequence we define, for $m \geq 0$,

$$\Delta^m b_s = \sum_{i=0}^{m} (-1)^i \binom{m}{i} b_{s+i}.$$

D. S. Mitrinović [4] has obtained some interesting extensions of (6) and Corollary 2, as for example:

Theorem 3. 1° *If* $1 \leq k \leq n-1$, $0 \leq \nu \leq k-1$, *then*

$$(\Delta^\nu e_{k-\nu})^2 - (\Delta^\nu e_{k-\nu+1})(\Delta^\nu e_{k-\nu-1}) \geq 0. \tag{7}$$

2° *If* $(-1)^p \Delta^p e_{k-\nu+1} > 0$, $1 \leq p \leq \nu$, *then* $(-1)^\nu \Delta^\nu e_{k-\nu} > 0$.

Proof. 1° follows from (4) applied to the polynomial $(x-1)^\nu \prod_{k=1}^{n} (a_k + x)$.

2° we prove by induction on ν. The case $\nu = 1$ is just Corollary 2. Let us assume the result for $\nu - 1$. By 1°

$$\Delta^{\nu-1} e_{k-\nu} \Delta^{\nu-1} e_{k-\nu+2} \leq (\Delta^{\nu-1} e_{k-\nu+1})^2,$$

which by the induction hypothesis is equivalent to

$$\frac{\Delta^{\nu-1} e_{k-\nu+2}}{\Delta^{\nu-1} e_{k-\nu+1}} \leq \frac{\Delta^{\nu-1} e_{k-\nu+1}}{\Delta^{\nu-1} e_{k-\nu}}. \tag{8}$$

But by hypothesis $(-1)^\nu \Delta^\nu e_{k-\nu+1} > 0$ is equivalent to the inequality $(-1)^{\nu-1} \Delta^{\nu-1} e_{k-\nu+2} > (-1)^{\nu-1} \Delta^{\nu-1} e_{k-\nu+1}$, or to

$$\frac{\Delta^{\nu-1} e_{k-\nu+2}}{\Delta^{\nu-1} e_{k-\nu+1}} > 1. \tag{9}$$

Inequalities (8) and (9) imply that

$$\frac{\Delta^{\nu-1} e_{k-\nu+1}}{\Delta^{\nu-1} e_{k-\nu}} > 1,$$

which is equivalent to $(-1)^\nu \Delta^\nu e_{k-\nu} > 0$, as was to be proved.

The basic idea in the proof of 1° in Theorem 3 can be used to obtain further results. For instance applying (4) to the polynomial $(x-\alpha) \prod_{k=1}^{n} (a_k + x)$, D. S. Mitrinović obtained the following result (see [4] and [5]):

$$4(e_{r-1}e_{r+1} - e_r^2)(e_{r-2}e_r - e_{r-1}^2) \geq (e_{r-1}e_r - e_{r-2}e_{r+1})^2.$$

Results analogous to Corollaries 1 and 2 and Theorem 3 can be obtained for symmetric means using (3) and (5), rather than (4) and (6). In particular if $r < s$, (5) implies that $p_{r-1}p_s \leq p_r p_{s-1}$, with equality if and only if $a_1 = \cdots = a_n$.

Using (2), this last inequality implies the following improvement of Corollary 1, due to J. Dougall [6]. If $r < s$, then

$$r(n - s + 1)\, e_r e_{s-1} - s(n - r + 1)\, e_{r-1} e_s > 0 .$$

The identity $\sum_{k=0}^{n} e_k x^k = \prod_{j=1}^{n} (1 + a_j x)$ suggests the following generalizations of (1) and (2) (see [7], [8] and [9]):

$$\sum_{k=0}^{+\infty} t_{k,s} x^k = \sum_{k=0}^{+\infty} T_n^{[k,s]}(a)\, x^k = \prod_{j=1}^{n} (1 + a_j x)^s \quad \text{for} \quad s > 0,$$

$$= \prod_{j=1}^{n} (1 - a_j x)^s \quad \text{for} \quad s < 0,$$

and $w_{k,s} = W_n^{[k,s]}(a) = \dfrac{t_{k,s}}{\binom{sn}{k}}$.

J. N. Whiteley [8] has shown that if $s > 0$, with $r < s$ if s is not an integer, $r \leq sn$ if s is an integer, then

$$w_{r,s}^2 - w_{r-1,s} w_{r+1,s} \geq 0. \tag{10}$$

The inequality is reversed if $s < 0$: since $w_{r,1} = p_r$, (10) is a significant generalization of (5).

Theorem 4. *If $1 < s < t < n$, then*

$$G_n(a) = G_n \leq p_t^{1/t} \leq p_s^{1/s} \leq A_n = A_n(a), \tag{11}$$

with equality if and only if $a_1 = \cdots = a_n$.

Proof. Since $p_1 = A_n$ and $p_n^{1/n} = G_n$, it is sufficient to consider the centre inequality.

By multiplying together inequality (5) in the form $(p_{r-1}p_{r+1})^r \leq p_r^{2r}$, $1 \leq r \leq s$, we get that $p_s^{1/s} \geq p_{s+1}^{1/(s+1)}$, which implies (11). The cases of equality are immediate.

Remark 1. In the same way inequality (10) implies an inequality similar to (11) for the Whiteley means [8].

Remark 2. It is possible to deduce (11) from the weaker inequality (6) [10]. This method can be used to extend (11) to the weighted symmetric means

$$P_n^{[r]}(a; q) = \frac{E_n^{[r]}(aq)}{E_n^{[r]}(q)}.$$

However, this expansion is not very satisfactory as it requires sequences a and q to be similarly ordered [10].

2.15.2 Inequalities of Rado-Popoviciu Type

Inequality (11) in 2.15.1 suggests the investigation of RADO-POPOVICIU results (see Remark in 2.14.6) for symmetric means; a fairly complete analogue for the POPOVICIU inequality is known ([10], [11] and [12]); results of the RADO type are incomplete, although recent results in this direction have been proved by D. S. MITRINOVIĆ and P. M. VASIĆ [13]. D. S. MITRINOVIĆ ([5] and [14]) has also obtained RADO type extensions to his results given in Theorem 3 in 2.15.1.

Let us introduce the following notations. If $a = (a_1, \dots, a_{n+q})$, then $\bar{a} = (a_1, \dots, a_n)$, $\tilde{a} = (a_{n+1}, \dots, a_{n+q})$, with a similar notation for the corresponding symmetric functions and means; thus $\bar{e}_r = E_n^{[r]}(\bar{a})$, $\tilde{p}_r = P_q^{[r]}(\tilde{a})$, etc.

The following identities can easily be proved [10]:

$$1^\circ \quad e_s = \sum_{t=0}^{s} \bar{e}_{s-t}\tilde{e}_t, \qquad s \le \min(n, q),$$

$$= \sum_{t=0}^{n+q-s} \bar{e}_{n-t}\tilde{e}_{s-n+t}, \qquad s > \max(n, q),$$

$$= \sum_{t=0}^{q} \bar{e}_{s-t}\tilde{e}_t, \qquad q < s \le n.$$

2° If $1 \le s \le n + q$, $u = \max(s - n, 0)$, $r = \min(s, q)$ and

$$\lambda(s, t) = \binom{n}{s-t}\binom{q}{t}\Big/\binom{n+q}{s}, \qquad 0 \le t \le s,$$

then

$$p_s = \sum_{t=u}^{r} \lambda(s, t)\, \bar{p}_{s-t}\tilde{p}_t. \tag{1}$$

3° In particular, if $a_{n+1} = \cdots = a_{n+q} = \beta$, (1) reduces to

$$p_s = \sum_{t=u}^{r} \lambda(s, t)\, \bar{p}_{s-t}\beta^t, \tag{2}$$

and if in addition, $a_1 = \cdots = a_n = \alpha$,

$$p_s = \sum_{t=u}^{r} \lambda(s, t)\, \alpha^{s-t}\beta^t. \tag{3}$$

Remark 1. The following is a simple application of (1). If $q = 1$, then $u = 0$, $r = 1$ and (1) becomes

$$p_s = \frac{n+1-s}{n+1} \bar{p}_s + \frac{s}{n+1} a_{n+1}\bar{p}_{s-1},$$

which is equivalent to

$$p_s - \bar{p}_s = \frac{s}{n+1}\bar{p}_{s-1}\left(a_{n+1} - \frac{p_s}{\bar{p}_{s-1}}\right).$$

But from (11) in 2.15.1

$$\frac{\bar{p}_s}{\bar{p}_{s-1}} \le \bar{A}_n$$

and so

$$p_s - \bar{p}_s \ge \frac{s}{n+1}\bar{p}_{s-1}(a_{n+1} - \bar{A}_n)\,.$$

Hence if a is increasing, $p_s - \bar{p}_s \ge 0$; thus $P_0^{[r]}(a), P_1^{[r]}(a), P_2^{[r]}(a), \ldots$ is increasing exactly when a is; further, it increases strictly if a does.

Theorem 1. *Let* $1 \le r < k \le n + q$, $u = \max(r - n, 0)$, $v = \min(r, q)$, $w = \max(k - n, 0)$, $x = \min(k, q)$.

1° *If* $v \le w$ *and* $r - u \le k - x$, *then*

$$\frac{p_r^{k/r}}{p_k} \ge \frac{\bar{p}_{r-u}^{(k-x)/(r-u)}}{\bar{p}_{k-x}} \cdot \frac{\tilde{p}_v^{w/v}}{\tilde{p}_w}\,; \tag{4}$$

2° *If* $v \le w$, *then*

$$\frac{p_r^{k/r}}{p_k} \ge \frac{\tilde{p}_v^{w/v}}{\tilde{p}_w}\,. \tag{5}$$

Proof. Rewrite (4) as

$$L = \frac{p_k^r}{\bar{p}_{k-x}^r \tilde{p}_w^r} \le \frac{p_r^k}{\bar{p}_{r-u}^{r(k-x)/(r-u)}\,\tilde{p}_v^{wr/v}} = R.$$

By (11) in 2.15.1

$$p_r^k = \left(\sum_{t=u}^{v} \lambda(r,t)\,\bar{p}_{r-t}\tilde{p}_t\right)^k \ge \left(\sum_{t=u}^{v} \lambda(r,t)\,\bar{p}_{r-u}^{(r-t)/(r-u)}\,\tilde{p}_v^{t/v}\right)^k. \tag{6}$$

(This inequality is strict unless $a_1 = \cdots = a_{n+q}$. However in certain cases this step is vacuous; in particular when $r = 1$, $k = n + q$, when all the means in (4) are either arithmetic or geometric means.)

It follows from (3) that this last expression is the k-th power of the r-th symmetric mean of $b_1, \ldots, b_{n+q}$, where $b_i = \bar{p}_{r-u}^{1/(r-u)}$, for $i = 1, \ldots, n$ and $b_j = \tilde{p}_v^{1/v}$, for $j = n+1, \ldots, n+q$. Hence by (11) in 2.15.1 and (3) again,

$$\begin{aligned} p_r^k &\ge \left(\sum_{t=w}^{x} \lambda(k,t)\,p_{r-u}^{(k-t)/(r-u)}\tilde{p}_v^{t/v}\right)^r \\ &= p_{r-u}^{r(k-x)/(r-u)}\tilde{p}_v^{wr/v}\left(\sum_{t=w}^{x} \lambda(k,t)\,p_{r-u}^{(x-t)/(r-u)}\,\tilde{p}_v^{(t-w)/v}\right)^r. \end{aligned}$$

(This inequality is strict unless $\bar{p}_{r-u}^v = \tilde{p}_v^{r-u}$; if the previous application of inequality (11) in 2.15.1 had not given a strict inequality, then neither can the present application. However if, as noted above, the previous application had been vacuous, then strict inequality could occur here.)

Using this last expression it easily follows that

$$R \geq \left(\sum_{t=w}^{x} \lambda(k, t)\, \bar{p}_{r-u}^{(x-t)/(r-u)}\, \tilde{p}_{v}^{(t-w)/v}\right)^{r} = S.$$

In a similar way, using (1),

$$p_k^r = \left(\sum_{t=w}^{x} \lambda(k, t)\, \bar{p}_{k-t}\, \tilde{p}_t\right)^r \leq \left(\sum_{t=w}^{x} \lambda(k, t)\, \bar{p}_{k-x}^{(k-t)/(k-x)}\, \tilde{p}_w^{t/w}\right)^r, \tag{7}$$

by (11) in 2.15.1, the inequality being strict unless $a_1 = \cdots = a_{n+q}$. So

$$p_k^r < \bar{p}_{k-x}^r\, \tilde{p}_w^r \left(\sum_{t=w}^{x} \lambda(k, t)\, \bar{p}_{k-x}^{(x-t)/(k-x)}\, \tilde{p}_w^{(t-w)/w}\right)^r,$$

which immediately gives

$$L \leq \left(\sum_{t=w}^{x} \lambda(k, t)\, \bar{p}_{k-x}^{(x-t)/(k-x)}\, \tilde{p}_w^{(t-w)/w}\right)^r = T.$$

But by (11) in 2.15.1, $T \leq S$, this inequality being strict unless $v = w$ and $r - u = k - x$, or $a_1 = \cdots = a_{n+q}$. This completes the proof of 1°.

The proof of 2° is similar except that when (11) in 2.15.1 is applied to the right-hand side of (6) and (7), it is applied to the second part of each term only, that is, to $\tilde{p}_t$.

Remark 2. In any particular application of this result the case of equality can be obtained from the above proof [15].

Remark 3. If $r = 1$, $k = n + q$, then (4) gives the following generalization of Popoviciu's inequality

$$\left(\frac{A_{n+q}}{G_{n+q}}\right)^{n+q} \geq \left(\frac{\bar{A}_n}{\bar{G}_n}\right)^n \left(\frac{\tilde{A}_q}{\tilde{G}_q}\right)^q,$$

with equality if and only if $\bar{A}_n = \tilde{A}_q$ (see [10] and [15]).

Remark 4. Taking $k = s + 1$, $q = 1$, (5) reduces to

$$\left(\frac{\bar{p}_r^{1/r}}{\bar{p}_s^{1/s}}\right)^s \leq \left(\frac{p_r^{1/r}}{p_{s+1}^{1/(s+1)}}\right)^{s+1}. \tag{8}$$

Remark 5. Inequalities similar to (8) have been obtained for elementary symmetric functions by D. S. Mitrinović and P. M. Vasić [12]. In particular if $q = 1$, $r < s$, $1 \leq s \leq n$ and $0 < \alpha \leq \beta$, then

$$\frac{(e_r)^\alpha}{(e_s)^\beta} \leq \frac{(\bar{e}_r)^\alpha}{(e_s)^\beta}.$$

We now quote a case of the application of the λ-method developed in 2.14.6.

Theorem 2. *If* $\lambda > 0$, $a = (a_1, \ldots, a_{n+1})$, *then for* $k = 2, 3, \ldots, n + 1$,

$$(9)\quad (n+1)\,(p_1 - \lambda p_k^{1/k}) \geq n\left(\bar{p}_1 - \left(\frac{(n+1)\,(k-1)}{nk}\right)\lambda^{k/(k-1)}\bar{p}_{k-1}^{1/(k-1)} - \frac{n+1-k}{nk}\,\frac{\bar{p}_k}{\bar{p}_{k-1}}\right),$$

with equality if and only if $a_{n+1} = \frac{n+1}{k}\lambda^{k/(k-1)}\bar{p}_{k-1}^{1/(k-1)} - \frac{n+1-k}{k}\,\frac{\bar{p}_k}{\bar{p}_{k-1}}$.

Proof. Put $x = a_{n+1}$ and

$$\begin{aligned} f(x) &= (n+1)\,(p_1 - \lambda p_k^{1/k}) \\ &= n\bar{p}_1 + x - (n+1)\,\lambda\left(\frac{n+1-k}{n+1}\,\bar{p}_k + \frac{k}{n+1}\,\bar{p}_{k-1}x\right)^{1/k}. \end{aligned}$$

Then f is defined provided $x \geq -\frac{n+1-k}{k}\,\frac{\bar{p}_k}{\bar{p}_{k-1}}$ and is easily seen to have a unique minimum at $x = \frac{n+1}{k}\lambda^{k/(k-1)}\bar{p}_{k-1}^{1/(k-1)} - \frac{n+1-k}{k}\,\frac{\bar{p}_k}{\bar{p}_{k-1}}$.

This completes the proof of Theorem 2 due to D. S. Mitrinović and P. M. Vasić [13].

Remark 6. In a similar way D. S. Mitrinović and P. M. Vasić [13] also proved the following Popoviciu type inequality for symmetric functions. If $\mu > 0$, then for $k = 2, 3, \ldots, n + 1$,

$$(10)\qquad \frac{(e_1+\mu)^k}{k^k e_k} \geq \frac{\left(\bar{e}_1 + \mu - \frac{\bar{e}_k}{\bar{e}_{k-1}}\right)^{k-1}}{(k-1)^{k-1}\bar{e}_{k-1}},$$

with equality if and only if $a_{n+1} = \frac{1}{k-1}\left(\mu + \bar{e}_1 - \frac{k\bar{e}_k}{\bar{e}_{k-1}}\right)$, ($\bar{e}_{n+1}$ being defined to be zero).

Setting $k = n + 1$, inequality (9) reduces to

$$(n+1)\,(A_{n+1} - \lambda\, G_{n+1}) \geq n\,(\bar{A}_n - \lambda^{(n+1)/n}\,\bar{G}_n).$$

Now, putting $k = n + 1$ and $\mu = \lambda\,(n + 1)$, inequality (10) becomes

$$\left(\frac{A_{n+1}+\lambda}{G_{n+1}}\right)^{n+1} \geq \left(\frac{\bar{A}_n + \frac{n+1}{n}\lambda}{\bar{G}_n}\right)^n.$$

It would be of interest to generalize (9), replacing p_1 by $p_s^{1/s}$ say; also to obtain a complete Rado analogue of Theorem 1.

The following result is due to D. S. Mitrinović [14].

Theorem 3. *Let* $a = (a_1, \ldots, a_{n+1})$ *and* $\bar{a} = (a_1, \ldots, a_n)$.

1° *If* $(-1)^p \Delta^p \bar{e}_{k-\nu} > 0$, $1 \le p \le \nu$, *then*

$$(-1)^\nu \Delta^\nu e_{k-\nu} > (-1)^\nu \Delta^\nu \bar{e}_{k-\nu}. \tag{11}$$

2° *If* $(-1)^\nu \Delta^\nu \bar{e}_{k-\nu-2} > 0$, $(-1)^\nu \Delta^\nu \bar{e}_{k-\nu-1} > 0$ *and* $(-1)^\nu \Delta^\nu \bar{e}_{k-\nu} > 0$, *then*

$$\begin{aligned}(\Delta^\nu e_{k-\nu})^2 &- (\Delta^\nu e_{k-\nu+1})(\Delta^\nu e_{k-\nu-1}) \\ &\ge (\Delta^\nu \bar{e}_{k-\nu})^2 - (\Delta^\nu \bar{e}_{k-\nu+1})(\Delta^\nu \bar{e}_{k-\nu-1}).\end{aligned} \tag{12}$$

Proof. The hypothesis implies by Theorem 3 in 2.15.1 that $(-1)^\nu \Delta^\nu e_{k-\nu-1} > 0$; this, together with the identity

$$\Delta^\nu e_{k-\nu} = a_{n+1} \Delta^\nu \bar{e}_{k-\nu-1} + \Delta^\nu \bar{e}_{k-\nu},$$

implies (11).

Put $x = a_{n+1}$ and

$$\begin{aligned}f(x) &= (\Delta^\nu e_{k-\nu})^2 - (\Delta^\nu e_{k-\nu+1})(\Delta^\nu e_{k-\nu-1}) \\ &= (x\Delta^\nu \bar{e}_{k-\nu-1} + \Delta^\nu \bar{e}_{k-\nu})^2 - (x\Delta^\nu \bar{e}_{k-\nu-2} + \Delta^\nu \bar{e}_{k-\nu-1})(x\Delta^\nu \bar{e}_{k-\nu} + \Delta^\nu \bar{e}_{k-\nu+1}).\end{aligned}$$

Simple calculations and (7) show that $f''(x) \ge 0$. In particular, these calculations give

$$f'(0) = \Delta^\nu \bar{e}_{k-\nu-1} \Delta^\nu \bar{e}_{k-\nu} - \Delta^\nu \bar{e}_{k-\nu-2} \Delta^\nu \bar{e}_{k-\nu+1} \ge 0,$$

by the hypotheses and (7).

Hence if $x \ge 0$, $f'(x) \ge 0$, which implies $f(x) \ge f(0)$, but this is just (12).

In particular if $\nu = 0$ the hypotheses are automatically satisfied and (12) reduces to $e_k^2 - e_{k-1}e_{k+1} \ge \bar{e}_k^2 - \bar{e}_{k-1}\bar{e}_{k+1}$.

2.15.3 Concavity of Certain Functions Involving the Elementary Symmetric Functions

The results in this Subsection due to Marcus and Lopes, McLeod and Whiteley (see [16], [17], [7] and [11]) are much deeper.

Let us introduce the following notations: if there exists a number λ such that $a_j = \lambda b_j$, $j \ge 1$, write $a \sim b$; if this is not the case, write $a \nsim b$; a_i' means the sequence a with the term a_i omitted.

Theorem 1. *If* $1 \le p \le r \le n$, *then*

$$\left(\frac{e_r(a+b)}{e_{r-p}(a+b)}\right)^{1/p} \ge \left(\frac{e_r(a)}{e_{r-p}(a)}\right)^{1/p} + \left(\frac{e_r(b)}{e_{r-p}(b)}\right)^{1/p}, \tag{1}$$

with equality if and only if $a \sim b$ *or* $r = 1$, $p = 1$. *In addition* $e_r(a)/e_{r-p}(a)$ *is an increasing function of each* a_i.

Proof. First suppose that (1) is known when $p = 1$. Then

$$\left(\frac{e_r(a+b)}{e_{r-p}(a+b)}\right)^{1/p} = \left(\prod_{j=1}^{p} \frac{e_{r-j+1}(a+b)}{e_{r-p}(a+b)}\right)^{1/p}$$

$$\geq \left(\prod_{j=1}^{p} \frac{e_{r-j+1}(a)}{e_{r-j}(a)} + \frac{e_{r-j+1}(b)}{e_{r-j}(b)}\right)^{1/p},$$

by (1) with $p = 1$. From this, using Theorem 10 of [2], we get

$$\left(\frac{e_r(a+b)}{e_{r-p}(a+b)}\right)^{1/p} \geq \left(\prod_{j=1}^{p} \frac{e_{r-j+1}(a)}{e_{r-j}(a)}\right)^{1/p} + \left(\prod_{j=1}^{p} \frac{e_{r-j+1}(b)}{e_{r-j}(b)}\right)^{1/p}$$

$$= \left(\frac{e_r(a)}{e_{r-p}(a)}\right)^{1/p} + \left(\frac{e_r(b)}{e_{r-p}(b)}\right)^{1/p}.$$

If $f(a_1, \ldots, a_n) = \dfrac{e_r(a)}{e_{r-p}(a)}$, then

$$f_i'(a_1, \ldots, a_n) = \frac{e_{r-p}(a)\, e_{r-1}(a_i') - e_r(a)\, e_{r-p-1}(a_i')}{e_{r-p}^2(a)}$$

$$= \frac{e_{r-p}(a_i')\, e_{r-1}(a_i') - e_r(a_i')\, e_{r-p-1}(a_i')}{e_{r-p}^2(a)}.$$

Hence by Corollary 1 of 2.15.1, $f_i'(a_1, \ldots, a_n) \geq 0$, which proves the last part of the theorem.

It remains to consider (1) with $p = 1$. We may clearly assume $r \geq 2$ and $a \nsim b$, and let us write

$$g_r(a) = \frac{e_r(a)}{e_{r-1}(a)}, \quad \varphi_r(a, b) = g_r(a+b) - g_r(a) - g_r(b).$$

Then

$$\varphi_2(a, b) = \frac{\sum_{i=1}^{n} (a_i B_n - b_i A_n)}{2\, A_n B_n (A_n + B_n)},$$

where $A_n = \sum_{i=1}^{n} a_n$, $B_n = \sum_{i=1}^{n} b_n$. This proves (1) in the case $r = 2$, so let us assume $r > 2$.

The following identities are easily demonstrated.

$$r e_r(a) = \sum_{i=1}^{n} a_i e_{r-1}(a_i'), \qquad e_r(a) = a_i e_{r-1}(a_i') + e_r(a_i'),$$

$$(n - r)\, e_r(a) = \sum_{i=1}^{n} e_r(a_i'), \qquad r e_r(a) = A_n e_r(a) - \sum_{i=1}^{n} a_i^2 e_{r-2}(a_i').$$

From these we easily deduce that

$$r g_r(a) = A_n - \sum_{i=1}^{n} \frac{a_i^2}{a_i + g_{r-1}(a_i')}$$

and

$$\varphi_r(a, b) = \frac{1}{r} \sum_{i=1}^{n} \left(\frac{a_i^2}{a_i + g_{r-1}(a_i')} + \frac{b_i^2}{b_i + g_{r-1}(b_i')} - \frac{(a_i + b_i)^2}{a_i + b_i + g_{r-1}(a_i' + b_i')} \right).$$

We now complete the proof by induction on r. By the induction hypothesis

$$g_{r-1}(a_i' + b_i') > g_{r-1}(a_i') + g_{r-1}(b_i')$$

unless $a_i' \sim b_i'$, in which case we have equality. Suppose first that for some i we have $a_i' \nsim b_i'$; then

$$(2)\ \varphi_r(a, b) > \frac{1}{r} \sum_{i=1}^{n} \frac{a_i^2}{a_i + g_{r-1}(a_i')} + \frac{b_i^2}{b_i + g_{r-1}(b_i')} - \frac{(a_i + b_i)^2}{a_i + b_i + g_{r-1}(a_i') + g_{r-1}(b_i')}$$

$$= \frac{1}{r} \sum_{i=1}^{n} \frac{(a_i g_{r-1}(b_i') - b_i g_{r-1}(a_i'))^2}{(a_i + g_{r-1}(a_i'))\,(b_i + g_{r-1}(b_i'))\,(a_i + b_i + g_{r-1}(a_i') + g_{r-1}(b_i'))}.$$

Now suppose that $a_i' \sim b_i'$ for every i when (2) is an equality; suppose in fact that $a_i' = \lambda_i b_i'$. Then

$$\left(a_i g_{r-1}(b_i') - b_i g_{r-1}(a_i')\right)^2 = (a_i - \lambda_i b_i)^2\, g_{r-1}(b_i')^2.$$

Hence since $a \nsim b$ this last expression is positive; and so in all cases $\varphi_r(a, b) > 0$.

Corollary 1. *If $1 \le r \le n$, then*

$$e_r(a + b)^{1/r} \ge e_r(a)^{1/r} + e_r(b)^{1/r},$$

with equality only when $r = 1$ or $a \sim b$.

Proof. This is just the case $r = p$ of Theorem 1.

Remark 1. This last result has been extended to Whiteley means [7];

$$(3)\qquad \left(T_n^{[k,s]}(a + b)\right)^{1/k} \ge \left(T_n^{[k,s]}(a)\right)^{1/k} + \left(T_n^{[k,s]}(b)\right)^{1/k}$$

if $s > 0$; the inequality being reversed if $s < 0$.

Remark 2. Since $E_n^{[r]}(a) = \sum_{\substack{i_1 + \cdots + i_n = r \\ i_j = 0,1}} a_1^{i_1} \cdots a_n^{i_n}$, this concept, and the corresponding mean is capable of considerable generalization [7]. One such generalization has been studied by R. F. Muirhead [2], p. 44, and more recently by G. Bekišev [18]. Their results are mainly concerned with the comparability of the various means they define.

One of the generalizations mentioned in Remark 2 is the complete elementary function, $c_r = c_r(a) = C_n^{[r]}(a) = \sum_{i_1+\cdots+i_n=r} a_1^{i_1}\cdots a_n^{i_n}$. A generalization of the r-th symmetric mean is: $D_r(a) = D_n^{[r]}(a) = \dfrac{C_n^{[r]}(a)}{\binom{r+n-1}{n-1}}$.

The case $s = -1$ of (3) shows that

$$\left(C_n^{[r]}(a+b)\right)^{1/r} \le \left(C_n^{[r]}(a)\right)^{1/r} + \left(C_n^{[r]}(b)\right)^{1/r}.$$

This result has also been proved by J. B. McLeod [17]. It would be of interest to obtain a more direct proof of this last inequality. J. B. McLeod [17], and K. V. Menon [19], have conjectured that a result similar to Theorem 1 should hold for the complete elementary functions. Certain other results have been obtained by K. V. Menon [19]; in particular the following analogue of Corollary 1, and Theorem 3 in 2.15.1.

Theorem 2. *If* $r < s$ *then*: 1° $c_r c_{s-1} > c_{r-1} c_s$, 2° $c_r^{1/r} > c_s^{1/s}$.

Proof. Let $a = (a_1, \ldots, a_{n+1})$ and $\bar{a} = (a_1, \ldots, a_n)$; then it follows easily that

$$c_r = \sum_{j=0}^{r} a_{n+1}^{r-j} \bar{c}_j \quad \left(c_0 = 1 \text{ and } \bar{c}_r = c_r(\bar{a})\right).$$

Hence

$$c_r c_{s-1} - c_{r-1} c_s = \sum_{j=0}^{r-1} a_{n+1}^{j} (\bar{c}_r \bar{c}_{s-1-j} - \bar{c}_{r-1-j} \bar{c}_s) + a_{n+1}^{r} \sum_{j=r}^{s-1} \bar{c}_{s-1-j}.$$

From this identity 1° can be obtained by a proof by induction on n, since clearly the result holds when $n = 1$.

2° follows from 1° in the usual way since 1° implies that

$$c_r^2 - c_{r-1} c_{r+1} > 0.$$

We also quote some other results from the paper [19] of K. V. Menon:

$$\frac{C_n^{[r]}(a+b)}{C_n^{[r-1]}(a+b)} \le \frac{C_n^{[r]}(a)}{C_n^{[r-1]}(a)} + \frac{C_n^{[r]}(b)}{C_n^{[r-1]}(b)}$$

for $r = 1$ and $r = 2$, $n = 2$ (in other cases a proof is not known);

$$D_{r-1}(a)\, D_{r+1}(a) - D_r(a)^2 \ge 0$$

for $r = 1, 2, 3$ (for $r \ge 4$ a proof is a not known);

$$D_{r-1}(a)\, D_{r+1}(a) - D_{r-2}(a)\, D_{r+2}(a) \ge 0$$

for $n = 2$ (for $n > 2$ the inequality has not yet been proved).

Concerning the topics of this Section see also papers [20] of G. Pólya, [21] of G. S. Mahajani, V. R. Thiruvenkatachar and V. D. Thawani, and [22] of G. S. Mahajani and V. R. Thiruvenkatachar.

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2.16 Steffensen's and Related Inequalities

This Section concerns a general inequality due to J. F. STEFFENSEN [1], published in 1918. The STEFFENSEN inequality does not appear in the book Inequalities by G. H. HARDY, J. LITTLEWOOD and G. PÓLYA, from 1934, which assembled almost all important inequalities.

The Jahrbuch über die Fortschritte der Mathematik did not review STEFFENSEN's paper [1], but G. SZEGÖ quoted the STEFFENSEN inequality in his review of papers [2] and [6] by T. HAYASHI.

The following result and its proof were given by STEFFENSEN [1].

Theorem 1. *Assume that two integrable functions f and g are defined on the interval (a, b), that f never increases and that $0 \leq g(t) \leq 1$ in (a, b). Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)\,g(t)\,dt \leq \int_{a}^{a+\lambda} f(t)\,dt, \tag{1}$$

where

$$\lambda = \int_{a}^{b} g(t)\,dt. \tag{2}$$

Proof. The second inequality of (1) may be derived as follows:

$$\begin{aligned}
&\int_{a}^{a+\lambda} f(t)\,dt - \int_{a}^{b} f(t)\,g(t)\,dt \\
&\qquad = \int_{a}^{a+\lambda} [1-g(t)]\,f(t)\,dt - \int_{a+\lambda}^{b} f(t)\,g(t)\,dt \\
&\qquad \geq f(a+\lambda)\int_{a}^{a+\lambda} [1-g(t)]\,dt - \int_{a+\lambda}^{b} f(t)\,g(t)\,dt \\
&\qquad = f(a+\lambda)\left[\lambda - \int_{a}^{a+\lambda} g(t)\,dt\right] - \int_{a+\lambda}^{b} f(t)\,g(t)\,dt \\
&\qquad = f(a+\lambda)\left[\int_{a}^{b} g(t)\,dt - \int_{a}^{a+\lambda} g(t)\,dt\right] - \int_{a+\lambda}^{b} f(t)\,g(t)\,dt \\
&\qquad = f(a+\lambda)\int_{a+\lambda}^{b} g(t)\,dt - \int_{a+\lambda}^{b} f(t)\,g(t)\,dt \\
&\qquad = \int_{a+\lambda}^{b} g(t)\,[f(a+\lambda) - f(t)]\,dt \\
&\qquad \geq 0.
\end{aligned}$$

The first inequality of (1) can be proved similarly. However, the second inequality of (1) implies the first.

Indeed, let $G(t) = 1 - g(t)$ and $\Lambda = \int_a^b G(t)\,dt$. Note that $0 \leq G(t) \leq 1$ if $0 \leq g(t) \leq 1$ in (a, b). Then

$$b - a = \lambda + \Lambda. \tag{3}$$

Suppose the second inequality of (1) holds. Then

$$\int_a^b f(t)\,G(t)\,dt \leq \int_a^{a+\Lambda} f(t)\,dt,$$

i.e.,

$$\int_a^b f(t)\,[1 - g(t)]\,dt \leq \int_a^{b-\lambda} f(t)\,dt,$$

i.e.,

$$\int_a^b f(t)\,dt - \int_a^{b-\lambda} f(t)\,dt \leq \int_a^b f(t)\,g(t)\,dt,$$

i.e.,

$$\int_{b-\lambda}^b f(t)\,dt \leq \int_a^b f(t)\,g(t)\,dt,$$

which is the first inequality of (1).

Remark 1. Roy O. Davies has communicated to us the following interesting proof of the second inequality of (1).

The function H, defined by

$$H(x) = \int_a^{a+\int_a^x g(t)\,dt} f(t)\,dt - \int_a^x f(t)\,g(t)\,dt,$$

is zero when $x = a$ and has a positive derivative:

$$H'(x) = f\left(a + \int_a^x g(t)\,dt\right) g(x) - f(x)\,g(x) \geq 0$$

since $a + \int_a^x g(t)\,dt \leq x$, because of the hypothesis $0 \leq g(t) \leq 1$, and thus

$$f\left(a + \int_a^x g(t)\,dt\right) \geq f(x)$$

as f is decreasing.

This holds for smooth functions, and can be extended to others by the usual approximations.

T. Hayashi, in [2], generalized inequality (1) slightly by taking the condition

$$0 \leq g(t) \leq A \quad (A \text{ is a constant } > 0)$$

instead of $0 \leq g(t) \leq 1$, and proved that

$$A \int_{b-\lambda}^{b} f(t)\, dt \leq \int_{a}^{b} f(t)\, g(t)\, dt \leq A \int_{a}^{a+\lambda} f(t)\, dt,$$

where

$$\lambda = \frac{1}{A} \int_{a}^{b} g(t)\, dt.$$

Some slight generalizations of the STEFFENSEN inequality were given by MEIDELL [3].

In [4], J. F. STEFFENSEN used the second inequality of (1) to derive a generalization of JENSEN's inequality for convex functions. He proved the following

Theorem 2. *If f is a convex function and x_k $(k = 1, \ldots, n)$ never decreases, and if c_k $(k = 1, \ldots, n)$ satisfies the conditions*

$$0 \leq \sum_{k=\nu}^{n} c_k \leq \sum_{k=1}^{n} c_k \qquad (\nu = 1, \ldots, n), \quad \textit{with} \quad \sum_{k=1}^{n} c_k > 0,$$

then

$$f\left(\frac{\sum_{k=1}^{n} c_k x_k}{\sum_{k=1}^{n} c_k}\right) \leq \frac{\sum_{k=1}^{n} c_k f(x_k)}{\sum_{k=1}^{n} c_k}. \tag{4}$$

This inequality is evidently more general than JENSEN's inequality [5] since the numbers c_k $(k = 1, \ldots, n)$ need not necessarily be positive.

A corresponding inequality for integrals was also given:

Theorem 3. *If f is a convex function, g never increases and h satisfies*

$$0 \leq \int_{\theta}^{1} h(x)\, dx \leq \int_{0}^{1} h(x)\, dx, \quad \textit{with} \quad 0 \leq \theta \leq 1, \quad \textit{and} \quad \int_{0}^{1} h(x)\, dx > 0,$$

then

$$f\left(\frac{\int_{0}^{1} h(x)\, g(x)\, dx}{\int_{0}^{1} h(x)\, dx}\right) \leq \frac{\int_{0}^{1} h(x)\, f(g(x))\, dx}{\int_{0}^{1} h(x)\, dx}. \tag{5}$$

T. HAYASHI in [6] obtained an upper bound for the right-hand sides in (4) and (5).

Assuming that $f(t) \to 0$, for $t \to +\infty$, and that f is integrable in $(0, +\infty)$, J. F. STEFFENSEN [7] applied (1) to deduce the following inequalities:

$$\sum_{k=1}^{+\infty}(-1)^k f(k\pi) < \int_0^{+\infty} f(t)\cos t\,dt < \sum_{k=0}^{+\infty}(-1)^k f(k\pi),$$

$$\sum_{k=0}^{+\infty}(-1)^k f\left(\left(k+\tfrac{1}{2}\right)\pi\right) < \int_0^{+\infty} f(t)\sin t\,dt < f(0) + \sum_{k=0}^{+\infty}(-1)^k f\left(\left(k+\tfrac{1}{2}\right)\pi\right).$$

J. F. STEFFENSEN also gave more precise inequalities in terms of

$$g(x) = \int_x^{x+1} f(t)\,dt.$$

It should be noticed that R. BELLMAN, in [8], refers to STEFFENSEN's paper [7] from 1947, as a source of inequality (1) but not to paper [1] from 1918, nor [4] from 1919, though this inequality was published for the first time in 1918. This is probably the reason why R. BELLMAN does not mention Theorems 2 and 3, in his paper [8], or monograph [9], published in cooperation with E. F. BECKENBACH.

R. BELLMAN [8] gave the following proof of STEFFENSEN's inequality (1) requiring f to be nonnegative.

Assuming that there does not exist an interval on which $f(t) = 0$, define the function u by the equality

$$\int_a^s f(t)\,g(t)\,dt = \int_a^{u(s)} f(t)\,dt, \tag{6}$$

whence $u(a) = a$, and

$$\int_a^{s+h} f(t)\,g(t)\,dt = \int_a^{u(s+h)} f(t)\,dt \quad (a < s+h < b). \tag{7}$$

Let $h > 0$. Then (6) and (7) yield

$$\int_s^{s+h} f(t)\,g(t)\,dt = \int_{u(s)}^{u(s+h)} f(t)\,dt.$$

This equality is valid only if $u(s+h) \geq u(s)$, i.e., if u is increasing. Since $0 \leq g(t) \leq 1$ and $f(t) \geq 0$ $(0 < t < b)$, we have

$$0 \leq \int_a^s f(t)\,g(t)\,dt \leq \int_a^s f(t)\,dt \quad (a < s < b). \tag{8}$$

From (6) and (8) it follows that

$$\int_a^{u(s)} f(t)\,dt \leq \int_a^s f(t)\,dt,$$

whence $u(s) \leq s$.

Starting from (6) and (7), we obtain

$$\left|\int_a^{u(s+h)} f(t)\,dt - \int_a^{u(s)} f(t)\,dt\right| = |u(s+h) - u(s)| \cdot \mu$$

$$= \left|\int_a^{s+h} f(t)\,g(t)\,dt - \int_a^{s} f(t)\,g(t)\,dt\right| = \left|\int_s^{s+h} f(t)\,g(t)\,dt\right|$$

$$\leq |h|\, f(a),$$

where

$$\inf f(t) \leq \mu \leq \sup f(t)$$

for $t \in [s, s+h]$ if $h > 0$ or $t \in [s+h, s]$ if $h < 0$.

This proves the continuity of u.

By differentiation equality (6) gives

$$f(u)\,\frac{du}{ds} = f(s)\,g(s) \quad \text{(almost everywhere)},$$

whence

$$\frac{du}{ds} = \frac{f(s)}{f(u)}\,g(s) \leq g(s),$$

taking account of the fact that $u(s) \leq s$ and that $f(s)$ is decreasing. Hence

$$\int_a^s du \leq \int_a^s g(s)\,ds,$$

i.e.,

(9) $$u(s) \leq a + \int_a^s g(s)\,ds.$$

(6) and (9) yield the right-hand inequality of (1).

Using the same procedure, R. Bellman, in [8], also established one of, as he points out, many possible generalizations of Steffensen's inequality. His generalization reads:

Theorem 4. *Let f be a nonnegative and monotone decreasing function in $[a, b]$ and $f \in L^p[a, b]$, and let g be a nonnegative and monotone increasing function in $[a, b]$ and $\int_a^b g(t)^q\,dt \leq 1$, where $p > 1$ and $1/p + 1/q = 1$. Then*

(10) $$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq \int_a^{a+\lambda} f(t)^p\,dt,$$

where

(11) $$\lambda = \left(\int_a^b g(t)\,dt\right)^p.$$

Remark 2. In connection with the formulation of Theorem 4 see [24].

Bellman's paper [8] from 1959 was preceded by a series of notes in which various inequalities, actually all of the Steffensen type, are established. These were given in the following.

G. Szegö [10] proved in 1950 the following result:

Theorem 5. *If $a_1 > a_2 > \cdots > a_{2m-1} > 0$ and f is a convex function in $[0, a_1]$, then*

$$\sum_{k=1}^{2m-1} (-1)^{k-1} f(a_k) \geq f\left(\sum_{k=1}^{2m-1} (-1)^{k-1} a_k\right).$$

In 1952, H. F. Weinberger [11] proved Theorem 5 for the function $x \mapsto f(x) = x^r$ $(r > 1)$, namely

Theorem 6. *If $a_1 \geq \cdots \geq a_n \geq 0$, then*

$$\sum_{k=1}^{n}(-1)^{k-1} a_k^r \geq \left(\sum_{k=1}^{n} (-1)^{k-1} a_k\right)^r \quad (r > 1).$$

In 1953, R. Bellman [12] proved a generalized version of Theorem 6:

Theorem 7. *Let $a_1 \geq \cdots \geq a_n \geq 0$ and let f be a convex function on $[0, a_1]$, with $f(0) \leq 0$. Then*

$$\sum_{k=1}^{n}(-1)^{k-1} f(a_k) \geq f\left(\sum_{k=1}^{n}(-1)^{k-1} a_k\right).$$

We note that the condition $f(0) \leq 0$ cannot be relaxed if there is an even number of terms, but may be omitted if n is odd, as given in Theorem 5.

E. M. Wright [13] in 1954 pointed out that Theorem 7 is a consequence of Theorem 108 in [14], p. 89, which reads:

Theorem 8 (Majorization theorem). *The conditions*

$$x_1 \geq \cdots \geq x_n, \; y_1 \geq \cdots \geq y_n,$$

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \textit{for} \quad k = 1, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

are necessary and sufficient in order that for every convex function f,

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i).$$

A proof of this theorem is given in Theorem 1 of 2.24.

M. Biernacki [15] in 1953 proved:

Theorem 9. *If* $a_1 \geq \cdots \geq a_n > 0$, $b_1 \geq \cdots \geq b_n > 0, \ldots, h_1 \geq \cdots \geq h_n > 0$ *and* $p > 1$, *then*

$$(a_1^p + b_1^p + \cdots + h_1^p) - (a_2^p + b_2^p + \cdots + h_2^p) + \cdots \pm (a_n^p + b_n^p + \cdots + h_n^p)$$
$$\geq (a_1 - a_2 + \cdots \pm a_n)^p + (b_1 - b_2 + \cdots \pm b_n)^p + \cdots + (h_1 - h_2 + \cdots \pm h_n)^p.$$

The inequality is reversed for $0 < p < 1$.

This theorem is an immediate consequence of Theorem 6.

Theorem 10. *Let* f *be a convex function for* $x > 0$ *and let* $f(0) = 0$. *Further, let*

$$0 \leq a_{i-1} \leq b_{i-1} \leq a_i \quad \text{for} \quad i = 2, \ldots, n,$$

and

$$0 \leq b_n, b_1 + \cdots + b_n \leq a_1 + \cdots + a_n.$$

Then

$$\big(f(a_1) + \cdots + f(a_n)\big) - \big(f(b_1) + \cdots + f(b_n)\big)$$
$$\geq f\big((a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)\big).$$

H. D. Brunk [16] proved in 1956 a general result having as a corollary:

Theorem 11. *Let* f *be a convex function on* $[a, b]$, *with* $f(0) \leq 0$. *Let* $b \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, *and let* $1 \geq h_1 \geq h_2 \geq \cdots \geq h_n \geq 0$. *Then*

$$\sum_{k=1}^{n} (-1)^{k-1} h_k f(a_k) \geq f\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right). \tag{12}$$

If $h_1 = \cdots = h_n = 1$, Theorem 11 reduces to Theorem 7.

I. Olkin [17] in 1959 proved:

Theorem 12. *Let* $1 \geq h_1 \geq \cdots \geq h_n \geq 0$ *and* $a_1 \geq \cdots \geq a_n \geq 0$. *Let* F *be a convex function on* $[0, a_1]$. *Then*

$$\left(1 - \sum_{k=1}^{n} (-1)^{k-1} h_k\right) F(0) + \sum_{k=1}^{n} (-1)^{k-1} h_k F(a_k) \tag{13}$$
$$\geq F\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right).$$

This theorem can be obtained from Theorem 11. Indeed, put $f(x) = F(x) - F(0)$. Then (12) becomes

$$\sum_{k=1}^{n} (-1)^{k-1} h_k\big(F(a_k) - F(0)\big) \geq F\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right) - F(0),$$

which is precisely inequality (13).

Now, we shall prove that inequality (13) can be derived from the second inequality of (1). First, for $t \in [0, a_1]$, put

$$g(t) = \lambda_k \quad \text{for} \quad a_{k+1} < t \leq a_k \ (k = 1, \dots, n), \quad \text{with} \quad a_{n+1} = 0,$$

where

$$\lambda_1 = h_1, \ \lambda_2 = h_1 - h_2, \dots, \quad \lambda_n = h_1 - h_2 + \cdots + (-1)^{n-1} h_n,$$

whence $0 \leq g(t) \leq 1$ in $[0, a_1]$.

Since

$$\lambda = \int_0^{a_1} g(t)\, dt = (a_1 - a_2)\lambda_1 + (a_2 - a_3)\lambda_2 + \cdots + (a_{n-1} - a_n)\lambda_{n-1} + a_n \lambda_n$$
$$= a_1 h_1 - a_2 h_2 + \cdots + (-1)^{n-1} a_n h_n,$$

we have

$$\lambda = \sum_{k=1}^{n} (-1)^{k-1} h_k a_k.$$

Now, if F is a function with increasing first derivative F', then $x \mapsto f(t) = -F'(t)$ is a decreasing function.

The functions g and f, defined above, satisfy all the conditions which allow the application of STEFFENSEN's inequality

$$\int_a^b f(t)\, g(t)\, dt \leq \int_a^{a+\lambda} f(t)\, dt.$$

Thus we have

$$\int_0^{a_1} F'(t)\, g(t)\, dt \geq \int_0^{\lambda} F'(t)\, dt,$$

i.e.,

$$\sum_{k=1}^{n} \big(F(a_{k+1}) - F(a_k)\big)\ (h_1 - h_2 + \cdots + (-1)^{k-1} h_k) \geq F(\lambda) - F(0).$$

This inequality is equivalent to inequality (13).

To the best of our knowledge, the following theorem was also proved for the first time by J. F. STEFFENSEN (see [18], p. 141):

Theorem 13. *Let g_1 and g_2 be functions defined on $[a, b]$ such that*

$$\int_a^x g_1(t)\, dt \geq \int_a^x g_2(t)\, dt \quad \textit{for all} \quad x \in [a, b]$$

and

$$\int_a^b g_1(t)\, dt = \int_a^b g_2(t)\, dt.$$

Let f be an increasing function on $[a, b]$, then

$$\int_a^b f(x)\, g_1(x)\, dx \le \int_a^b f(x)\, g_2(x)\, dx.$$

If f is a decreasing function on $[a, b]$, then

$$\int_a^b f(x)\, g_1(x)\, dx \ge \int_a^b f(x)\, g_2(x)\, dx.$$

Proof. Put $g(x) = g_1(x) - g_2(x)$ and $G(x) = \int_a^x g(t)\, dt$. Then, by the above hypothesis,

$$G(x) \ge 0 \quad (a \le x \le b) \qquad \text{and} \qquad G(a) = G(b) = 0.$$

Using the STIELTJES integral, we get

$$\begin{aligned} \int_a^b f(t)\, g(t)\, dt &= \int_a^b f(t)\, dG(t) \\ &= f(t)\, G(t) \Big|_a^b - \int_a^b G(t)\, df(t) \\ &= -\int_a^b G(t)\, df(t). \end{aligned}$$

This proves Theorem 13.

M. MARJANOVIĆ, in [19], considered the above inequality as a sp ecia case of a general inequality due to K. FAN and G. G. LORENTZ [20] and used it to give the following short proof of (1).

Let $g_2(x) = g(x)$, $\lambda = \int_a^b g(x)\, dx$ and $g_1(x) = 1$ for $x \in [a, a + \lambda]$ and $g_1(x) = 0$ for $x \in [a + \lambda, b]$.

Then, we have

$$\int_a^{a+\lambda} f(x)\, dx = \int_a^b f(x)\, g_1(x)\, dx \ge \int_a^b f(x)\, g(x)\, dx,$$

which proves the second inequality in (1). One similarly derives the first inequality in (1).

P. VERESS, in [21], used the technique of the STIELTJES integration to obtain an inequality containing the inequality in Theorem 13, as well as its discrete form.

Now, we shall state some results of Z. CIESIELSKI [22], related to Theorems 2 and 3 of STEFFENSEN. Apparently CIESIELSKI was unaware of STEFFENSEN's results.

Theorem 14. *Let (p_i) denote a sequence of real numbers such that*

$$\sum_{i=1}^{k} p_i \geq 0 \quad \textit{for} \quad k = 1, \ldots, n \quad \textit{and} \quad \sum_{i=1}^{n} |p_i| > 0.$$

Let $x_i \in [0, a]$ (where a is a positive constant) for $i = 1, \ldots, n$ and let $x_1 \geq \cdots \geq x_n$. Further, let f and f' be convex functions in $[0, a]$ and let $f(0) \leq 0$. Then

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} |p_i|}\right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} |p_i|}.$$

Theorem 15. *Let the function g be nonincreasing in $[\alpha, \beta]$ and let $a \geq g(t) \geq 0$ in $[\alpha, \beta]$. Let f and f' be convex in $[0, a]$ and let $f(0) \leq 0$. Further, let p be a function integrable in the Lebesgue sense in $[\alpha, \beta]$, such that*

$$\int_{\alpha}^{x} p(t)\, dt \geq 0 \quad \textit{for} \quad x \in [\alpha, \beta] \quad \textit{and} \quad \int_{\alpha}^{\beta} |p(t)|\, dt > 0.$$

Then

$$f\left(\frac{\int_{\alpha}^{\beta} p(t)\, g(t)\, dt}{\int_{\alpha}^{\beta} |p(t)|\, dt}\right) \leq \frac{\int_{\alpha}^{\beta} p(t)\, f(g(t))\, dt}{\int_{\alpha}^{\beta} |p(t)|\, dt}.$$

In the same paper analogous results were given for functions of two variables and applications were made to establish generalizations of ČEBYŠEV's and BIERNACKI's inequalities [15].

Remark 3. R. P. BOAS, inspired by a paper of D. S. MITRINOVIĆ [see: Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **247–273**, 1–14 (**1969**)], has proved that Theorems 14 and 15 are corollaries of Theorems 2 and 3 in a stronger form: all that is needed on f is that f is convex and $f(0) \leq 0$. This is a private communication, which has not yet been published. P. R. BOAS has also obtained some interesting results concerning Theorems 2 and 3.

R. APÉRY in his short note [23] which contains no references, proved a variant of STEFFENSEN's inequality. His result reads:

Theorem 16. *Let f be a decreasing function in $(0, +\infty)$. Let g be a measurable function in $[0, +\infty)$ such that $0 \leq g(x) \leq A$ (A is a constant $\neq 0$). Then*

$$\int_{0}^{+\infty} f(x)\, g(x)\, dx \leq A \int_{0}^{\lambda} f(x)\, dx,$$

where

$$\lambda = \frac{1}{A} \int_{0}^{+\infty} g(x)\, dx.$$

R. Apéry, in his elegant proof, starts from the identity

$$\int_0^{+\infty} f(x)\,g(x)\,dx = A\int_0^{\lambda} f(x)\,dx - \int_0^{\lambda} [A - g(x)]\,[f(x) - f(\lambda)]\,dx$$
$$- \int_{\lambda}^{+\infty} g(x)\,[f(\lambda) - f(x)]\,dx.$$

Steffensen's inequalities (1) follow immediately if we use Apéry's idea, namely if we start from the identities

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)\,g(t)\,dt$$
$$= \int_a^{a+\lambda} [f(t) - f(a+\lambda)]\,[1 - g(t)]\,dt + \int_{a+\lambda}^b [f(a+\lambda) - f(t)]\,g(t)\,dt$$

and

$$\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt$$
$$= \int_a^{b-\lambda} [f(t) - f(b-\lambda)]\,g(t)\,dt + \int_{b-\lambda}^b [f(b-\lambda) - f(t)]\,[1 - g(t)]\,dt.$$

Finally, we notice that E. K. Godunova and V. I. Levin [24] have recently obtained a general result which contains Steffensen's inequality (1). This, once again, affirms the importance of this inequality which arises from various other inequalities.

We believe that it would be more appropriate to speak in [24] of a generalization of Steffensen's inequality, rather than a general inequality which contains Steffensen's inequality as a particular case.

Also related to the Steffensen inequality are references [25] and [26].

Remark 4. Though the Steffensen inequality is not included in the source book for inequalities [14], in recent time it is cited even in books dedicated to university studies, as for example, in [27], p. 83, and [28], p. 50. It should be noted that Steffensen's inequality can be found in Bourbaki [29]. J. Dieudonné [28], p. 50, gave the Steffensen inequality in the following form:

Let f and g be two piecewise continuous functions on $[a, b]$ such that f is decreasing and $0 \leq g(t) \leq 1$ on $[a, b]$. Putting

$$\lambda = \int_a^b g(t)\,dt,$$

we have

$$\int_{b-\lambda}^b f(t)\,dt \leq \int_a^b f(t)\,g(t)\,dt \leq \int_a^{a+\lambda} f(t)\,dt,$$

with equality holding only if f is a constant on $[a, b]$ or if g is equal to 0 or to 1 in all its points of continuity.

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2.17 Schur's Inequality

Theorem 1. *If x, y, z are positive numbers and if λ is real, then*

$$x^\lambda(x-y)(x-z)+y^\lambda(y-z)(y-x)+z^\lambda(z-x)(z-y)\geq 0, \tag{1}$$

with equality if and only if $x=y=z$.

This is I. Schur's inequality (see [1], p. 64; [2], p. 217, and [3]–[7]). Among various generalizations, the widest seems to be that due to U. C. Guha [8], which may be formulated as follows.

Theorem 2. *If a, b, c, u, v, w are positive real numbers, and*

$$a^{\frac{1}{p}}+c^{\frac{1}{p}}\leq b^{\frac{1}{p}}, \tag{2}$$

$$u^{\frac{1}{p+1}}+w^{\frac{1}{p+1}}\geq v^{\frac{1}{p+1}}, \tag{3}$$

then, if $p>0$,

$$ubc-vca+wab\geq 0. \tag{4}$$

If $-1<p<0$, the inequalities (3) *and* (4) *must be reversed; if $p<-1$ the inequalities* (2) *and* (3) *must be reversed. In each case there is equality in* (4) *if and only if there is equality in* (2) *and* (3) *and also*

$$\frac{a^{p+1}}{u^p}=\frac{b^{p+1}}{v^p}=\frac{c^{p+1}}{w^p}.$$

Proof. When $p>0$, it follows from Hölder's inequality that

$$\left[a^{\frac{1}{p+1}}(uc)^{\frac{1}{p+1}}+c^{\frac{1}{p+1}}(wa)^{\frac{1}{p+1}}\right]^{p+1}\leq(uc+wa)\left(a^{\frac{1}{p}}+c^{\frac{1}{p}}\right)^p,$$

in other words

$$ac\left(u^{\frac{1}{p+1}}+w^{\frac{1}{p+1}}\right)^{p+1}\leq(uc+wa)\left(a^{\frac{1}{p}}+c^{\frac{1}{p}}\right)^p,$$

and (4) follows immediately by (2) and (3). The other cases are dealt with similarly, and the conditions for equality can be read off from those for Hölder's inequality.

Derivation of Theorem 1. We may suppose without loss of generality that $0 \leq z \leq y \leq x$. In Theorem 2 put $p = 1$, $a = y - z$, $b = x - z$, $c = x - y$, $u = x^\lambda$, $v = y^\lambda$, $w = z^\lambda$, then (4) becomes (1). Condition (2) holds with equality, and (3) holds because

$$u^{\lambda/2} \geq v^{\lambda/2} \text{ or } w^{\lambda/2} \geq v^{\lambda/2}$$

according as $\lambda \geq 0$ or $\lambda \leq 0$.

SCHUR's inequality was partially generalized in another direction by K. S. AMUR [9], whose result (for the case $n = 5$) was completed as follows by A. OPPENHEIM and ROY O. DAVIES [10].

Theorem 3. *The necessary and sufficient conditions on a set of n (≥ 3) real constants $a_1, \ldots, a_n$ for the inequality*

$$\Sigma = \sum_{i=1}^{n} a_i (x_i - x_1) \cdots (x_i - x_{i-1}) (x_i - x_{i+1}) \cdots (x_i - x_n) \geq 0 \tag{5}$$

to hold for all real numbers $x_1, \ldots, x_n$ satisfying $x_1 \geq \cdots \geq x_n$ are that

for $n = 3$:

$$a_1 \geq 0, \ a_2 \leq (a_1^{1/2} + a_3^{1/2})^2, \ a_3 \geq 0,$$

for $n \geq 4$:

$$a_2 \leq a_1, (-1)^n (a_{n-1} - a_n) \geq 0, \ (-1)^{k+1} a_k \geq 0, \tag{6}$$

where $1 \leq k \leq n$, $k \neq 2, n - 1$.

(For $n = 5$, $a_1 = \cdots = a_5 = 1$, and $x_4 = x_5 = 0$, (5) becomes SCHUR's inequality for $\lambda = 2$.)

Proof for $n = 3$. *Sufficiency.* This is immediate from the identity

$$\begin{aligned} \Sigma = {} & [a_1^{1/2} (x_1 - x_2) - a_3^{1/2} (x_2 - x_3)]^2 \\ & + [(a_1^{1/2} + a_3^{1/2})^2 - a_2] (x_1 - x_2) (x_2 - x_3), \end{aligned} \tag{7}$$

which incidentally also shows when equality can occur in (5).

Necessity. Putting $x_1 > x_2 = x_3$ or $x_1 = x_2 > x_3$ in (5) gives the necessity of the conditions $a_1 \geq 0$, $a_3 \geq 0$; that of the other condition then follows from (7) upon choosing $x_1 - x_2 : x_2 - x_3 = a_3^{1/2} : a_1^{1/2}$.

Proof for $n \geq 4$. *Sufficiency.* If the conditions are satisfied, then the first two terms in Σ have the sum

$$\begin{aligned} & a_1 (x_1 - x_2) \cdots (x_1 - x_n) + a_2 (x_2 - x_1) \cdots (x_2 - x_n) \\ & \qquad = (x_1 - x_2) [a_1 (x_1 - x_3) \cdots (x_1 - x_n) - a_2 (x_2 - x_3) \cdots (x_2 - x_n)], \end{aligned}$$

which is nonnegative if $a_1 \geq 0 \geq a_2$, or $a_1 \geq a_2 \geq 0$ and $x_1 - x_3 \geq x_2 - x_3 \geq 0, \ldots, x_1 - x_n \geq x_2 - x_n \geq 0$. A similar argument applies to the last two terms in Σ; and each of the other terms is obviously nonnegative if (6) holds, since (with $x_r - x_r$ omitted) $(x_r - x_1) \cdots (x_r - x_n) \geq 0$ if r is odd, and ≤ 0 if r is even, that is, it has the same sign as a_r.

Necessity. If (6) implies (5), then putting $x_1 > x_2 > x_3 = \cdots = x_n$ and dividing by $x_1 - x_2$ yields that

$$a_1(x_1 - x_3) \cdots (x_1 - x_n) - a_2(x_2 - x_3) \cdots (x_2 - x_n) \geq 0,$$

and by continuity this also holds for $x_1 \geq x_2 \geq x_3 = \cdots = x_n$. Putting $x_1 > x_2 = x_3$ now gives the necessity of the condition $a_1 \geq 0$, while putting $x_1 = x_2 > x_3$ gives that of the condition $a_1 \geq a_2$. A similar argument applies to a_{n-1} and a_n; and the condition on a_r for $3 \leq r \leq n - 2$ is obtained at once upon putting

$$x_1 = \cdots = x_{r-1} > x_r > x_{r+1} = \cdots = x_n.$$

A. Oppenheim and Roy O. Davies [10] also noted the following as an immediate corollary.

Theorem 4. *The necessary and sufficient conditions on a set of n (≥ 3) real constants $a_1, \ldots, a_n$ for the inequality* (5) *to hold for all real numbers $x_1, \ldots, x_n$ are that*

for $n = 3$:

$$0 \leq a_i \leq a_j^{1/2} + a_k^{1/2} \quad \text{for every permutation } ijk \text{ of } 123,$$

for $n = 4$ and $n = 6$:

$$a_1 = \cdots = a_n = 0,$$

for $n = 5$:

$$a_1 = \cdots = a_5 \geq 0.$$

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2.18 Turán's Inequalities

Let $z_1, \ldots, z_n$ and $b_1, \ldots, b_n$ be arbitrary complex numbers such that $z_k \neq 0$ $(k = 1, \ldots, n)$ and $b_1 + \cdots + b_n \neq 0$. For any integer $m \geq -1$ we define

$$Q = \max_{r=m+1,\ldots,m+n} \frac{|b_1 z_1^r + \cdots + b_n z_n^r|}{|b_1 + \cdots + b_n| \min\limits_{1 \leq k \leq n} |z_k|^r}.$$

In his book [1] P. Turán was interested in finding a lower estimate for Q which is independent of the b_k's and z_k's. P. Turán ([1], p. 40) found that

$$(1) \qquad Q \geq \left(\frac{n}{2e(m+n)}\right)^n.$$

This estimate was improved by I. Dancs [2] to

$$(2) \qquad Q \geq \frac{1}{2e}\left(\frac{n}{2e(m+n)}\right)^{n-1}.$$

V. Sós-Turán and P. Turán [3] proved that the constant $2e$ in (1) could not be replaced by a number less than $4/\pi$.

The best possible estimate was found by E. Makai [4] and somewhat later by N. G. de Bruijn [5]:

$$(3) \qquad Q \geq \left(\sum_{k=0}^{n-1} 2^k \binom{m+k}{k}\right)^{-1}.$$

It seems that estimate (1) (or (2)) is more suitable for applications than (3).

From the fact that the estimate (3) is the best possible, from (1) we deduce that

$$\sum_{k=0}^{n-1} 2^k \binom{m+k}{k} \leq \left(\frac{2e(m+n)}{n}\right)^n.$$

Result (1) is known as "the first main theorem of Turán". We quote an application of this theorem (see P. Turán: Acta Math. Acad. Sci. Hung. **20**, 357–360 (1969)):

"Let $a > b$ and $c > 0$, and let $y(t)$ be an arbitrary complex-valued solution of the differential equation

$$(4) \qquad y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0,$$

where the a_k's are arbitrary complex constants. Let all the zeros of the corresponding characteristic equation of (4) lie in the half-plane $\operatorname{Re} z \geq L$. Then the inequality

$$(5) \qquad \int_a^{a+c} |y(t)|^2\,dt \geq \left(\frac{c}{2e(2a-2b+c)}\right)^{n^2} e^{-2(a-b+c)|L|} \int_b^{b+c/2} |y(t)|^2\,dt$$

holds."

Another theorem of P. Turán ([1], p. 47) is the following:

Let $z_1, \ldots, z_n$ and $b_1, \ldots, b_n$ be arbitrary complex numbers such that $1 = |z_1| \geq |z_2| \geq \cdots \geq |z_n|$. Then there exists a constant A, independent of m and n, such that

$$\max_{r=m+1,\ldots,m+n} \left|\sum_{k=1}^{n} b_k z_k^r\right| \geq \left(\frac{n}{A\,(m+n)}\right)^n \min_{r=1,\ldots,n} |b_1 + \cdots + b_r|. \tag{6}$$

V. Sós-Turán and P. Turán [3] have proved that $1.32 \leq A \leq 24$. S. Uchiyama [6] has shown that $e \leq A \leq 8e$. E. Makai [4] proved that $A \geq 2e/\log 2$. Finally, E. Makai [7] showed by an ingenious example that $A \geq 4e$.

In a previous paper [8] E. Makai proved that the exponent n in (6) can be replaced by $n - 1$.

In [8] E. Makai established the following inequality:

If $z_0 = b_0 = 1$ and $b_1, \ldots, b_n$ are arbitrary complex numbers, then

$$\inf_{b_k} \max_{r=m,\ldots,m+n} |b_0 z_0^r + b_1 z_1^r + \cdots + b_n z_n^r| \geq \frac{|z_1 - 1|\,|z_2 - 1| \cdots |z_n - 1|}{|s_0| + |s_1| + \cdots + |s_n|},$$

where the s_k's are the elementary symmetric functions of $z_1, \ldots, z_n$.

Improving a theorem of Turán ([1], p. 27), F. V. Atkinson [9] proved the following result:

If $z_1, \ldots, z_{n-1}$ are arbitrary complex numbers such that $|z_k| \leq 1$ $(k = 1, \ldots, n-1)$ and $z_n = 1$, then

$$\max_{r=1,\ldots,n} |z_1^r + \cdots + z_n^r| > \frac{1}{6}. \tag{7}$$

In a technical report [10] F. V. Atkinson has shown that the lower bound 1/6 could be raised to 1/3, but this result is not the best possible. In [11] V. F. Atkinson has given an improved version of the results from [10]. He demonstrated, in fact, that the lower bound $\pi/8$ is valid for a large range of values n, and that this bound is good at least for $n < 1.6 \cdot 10^3$. For large values of n, Atkinson obtained a lesser lower bound, the root of a given transcendental equation.

Let us denote

$$M_n = \min_{|z_k| \leq 1} \max_{r=1,\ldots,n} |z_1^r + \cdots + z_n^r|, \tag{8}$$

where the minimum is taken over all $z_1, \ldots, z_{n-1}, z_n$ satisfying $|z_k| \leq 1$ $(k = 1, \ldots, n-1)$ and $z_n = 1$. It is easy to show that $M_2 = \sqrt{3 - \sqrt{5}}$ and that this minimum is attained only for

$$z_1 = \frac{\sqrt{5} - 1}{2} \exp\left(\pm \frac{2\pi i}{3}\right).$$

It was proved by S. Uchiyama [12] that

$$M_n > (1-\varepsilon)\frac{\log\log n}{\log n}$$

for arbitrarily small $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

J. Ławrinowicz [13] proved that M_n in (8) can be defined also by

$$M_n = \min_{z_k} \max_{r=1,\dots,n} |z_1^r + \cdots + z_n^r|,$$

where $z_n = 1$ and the minimum is taken over all complex numbers $z_1, \dots, z_{n-1}$.

An application: if A is an $n \times n$ matrix, then all its eigenvalues are contained in the disc

$$|z| \le 3 \max_{r=1,\dots,n} |\operatorname{tr} A^r|^{1/r}. \tag{9}$$

J. W. S. Cassels [14] considered the case, where r ranges over the values $1, \dots, 2n-1$. He proved that

$$\max_{r=1,\dots,2n-1} |z_1^r + \cdots + z_n^r| \ge 1, \quad \text{if} \quad \max_{k=1,\dots,n} |z_k| = 1. \tag{10}$$

As an application of (10), the disc (9) can be replaced by

$$|z| \le \max_{r=1,\dots,2n-1} |\operatorname{tr} A^r|^{1/r}.$$

In [15] P. Turán proved the following proposition:

Let $z_1, \dots, z_n$ be complex numbers such that $\max |z_k| = 1$. Let

$$M_n = \min_{z_k} \max_{r=1,\dots,n} |s_r|,$$

where the minimum is taken over all z_k satisfying the additional restrictions $s_1 = s_2 = \cdots = s_{[n/2]} = 0$ and $s_r = z_1^r + \cdots + z_n^r$.

Then, for $n = 2m+1$,

$$M_{2m+1} \le \left(\sum_{r=m+1}^{2m+1} \frac{1}{r}\right)^{-1},$$

with equality if and only if z_k are the roots of the equation

$$\left(\sum_{r=m+1}^{2m+1} \frac{1}{r}\right) z^{2m+1} - \sum_{r=m+1}^{2m+1} \frac{z^{2m+1-r}}{r} = 0.$$

Let $z_1, \dots, z_n$ be arbitrary complex numbers not all zeros and

$$s_k = z_1^k + \cdots + z_n^k \qquad (k = 1, 2, \dots).$$

Let further $m = \max_{1 \le j \le n} |z_j|$ and $M = \max_{1 \le k \le n} |s_k/n|^{1/k}$.

J. D. Buckholtz proved in [16] that

$$\frac{\sqrt{2}-1}{2} < \frac{M}{m} \le 1,$$

where both bounds are the best possible.

Another result of J. D. BUCKHOLTZ [17], with the same notations, is: If $\lambda > -1$ and n is an integer greater than $(6 + 3\lambda)^3$, then

$$\sum_{k=1}^{n} k^{\lambda} |s_k| > \frac{1+\lambda}{2} \log n,$$

where the constant $(1 + \lambda)/2$ is the best possible.

The papers [18] and [19] are also related to problems exposed in this Section.

Remark. J. M. GEYSEL, R. TIJDEMAN and A. J. VAN DER POORTEN have also obtained some interesting extensions of certain results exposed above [see detailed reviews in Zentralblatt für Mathematik **179**, 70—73 (1970)].

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2.19 Benson's Method

D. C. BENSON has given an elementary method for proving inequalities and using it deduced a large number of known classical inequalities. This method is contained in the following:

1° Start from a suitably chosen algebraic inequality and use it to show that an expression which contains a function together with its derivatives is nonnegative.

2° To both sides of the inequality add something which is, for example, an exact derivative.

3° Integrate both sides of the inequality to obtain the wanted inequality.

The main results obtained by D. C. BENSON by this method are contained in the following two theorems.

Theorem 1. *Let $u(x)$, $P(u, x)$, $G(u, x)$ be continuously differentiable functions and let $P(u, x) > 0$. Then, the following inequality holds*

$$\int_a^b \left(Pu'^{2n} + (2n-1)\, P^{\frac{-1}{2n-1}} G_u^{\frac{2n}{2n-1}} + 2nG_x\right) dx \tag{1}$$

$$\geq 2n\big(G(u(b), b) - G(u(a), a)\big),$$

where $G_u = \frac{\partial}{\partial u} G(u, x)$, $G_x = \frac{\partial}{\partial x} G(u, x)$.

Equality holds if and only if $u' = (G_u/P)^{\frac{1}{2n-1}}$.

Remark. For the sake of brevity in this Section we write, for example, P^{-1} for $1/P$.

Proof. 1° Start from the following algebraic inequality

$$x^{2n} - 2nx + 2n - 1 \geq 0, \tag{2}$$

which is true for every real x and $n = 1, 2, \ldots$ (see 3.3.24), equality holding if and only if $x = 1$.

Put $x = u'(P/G_u)^{\frac{1}{2n-1}}$ in (2). After multiplying by $P^{-\frac{1}{2n-1}} G_u^{\frac{2n}{2n-1}}$ which is positive, we get

$$Pu'^{2n} + (2n-1)\, P^{\frac{-1}{2n-1}} G_u^{\frac{2n}{2n-1}} - 2nu'G_u \geq 0.$$

2° Add to both sides of the last inequality the following expression: $2n \frac{d}{dx} (G(u(x), x))$.

3° Integrate from a to b. Inequality (1) results, equality holding if and only if $x = 1$ in (2), i.e., $u' = (G_u/P)^{\frac{1}{2n-1}}$.

Theorem 2. *Let u be twice continuously differentiable, $P(u', u, x)$, $G(u', u, x)$ continuously differentiable functions and $P(u', u, x) > 0$. Then the following inequality holds*

$$\int_a^b (Pu''^2 + G_{u'}^2 P^{-1} + 2u'G_u + 2G_x)\,dx \tag{3}$$

$$\geq 2G(u'(b), u(b), b) - 2G(u'(a), u(a), a),$$

where the same abbreviations are used as in Theorem 1.

Equality holds if and only if $u'' = G_{u'}P^{-1}$.

Theorems 1 and 2 also hold under less restrictive conditions.

Inequalities (1) and (3) contain a number of known inequalities. We shall show how WIRTINGER's inequality can be obtained from (1).

Let u be a continuously differentiable periodic function of period 2π and let $m = \inf u(x)$, $M = \sup u(x)$. Further, let $m = u(x_1)$, $M = u(x_2)$, $(0 \leq x_1 - x_2 \leq 2\pi)$. Putting in (1) $P(u, x) = 1$, $n = 1$ and assuming that $x_2 < x_1$,

$$G(u, x) = -\int_M^u ((M-u)(u-m))^{1/2}\,du \qquad (x_2 \leq x \leq x_1),$$

$$= \int_M^u ((M-u)(u-m))^{1/2}\,du \qquad (x > x_1 \text{ or } x < x_2),$$

we get the following two inequalities

$$\int_{x_2}^{x_1} (u'^2 + (M-u)(u-m))\,dx \geq -2\int_M^m ((M-u)(u-m))^{1/2}\,du$$

$$= \frac{\pi(M-m)^2}{4}$$

and

$$\int_{x_1}^{x_2+2\pi} (u'^2 + (M-u)(u-m))\,dx \geq -2\int_M^m ((M-u)(u-m))^{1/2}\,du$$

$$= \frac{\pi(M-m)^2}{4}.$$

From those inequalities we obtain

$$\int_0^{2\pi} \left(u'^2 - \left(u - \frac{M+m}{2}\right)^2\right) dx \geq 0.$$

If additionally we assume $\int_0^{2\pi} u\,dx = 0$, we have

$$\int_0^{2\pi} (u'^2 - u^2)\,dx \geq \frac{\pi(M+m)^2}{2}$$

Since the right-hand side of this inequality is always positive, as a special case we obtain WIRTINGER's inequality (see 2.23.1).

Let us now state a useful special case of (1) which can then be applied to prove WEYL's inequality.

Let u and g be continuously differentiable. Then

$$\int_a^b \left(u'^2 + \left(g'(x) + g(x)^2\right) u^2\right) dx \geq u(b)^2 g(b) - u(a)^2 g(a). \tag{4}$$

Equality holds if and only if $u' = ug(x)$.

Inequality (4) can be directly obtained from (1) by putting $n = 1$, $P = 1$ and $G(u, x) = \frac{1}{2} u^2 g(x)$.

WEYL's inequality can be proved in the following way. Replace $g(x)$ in (4) by $\lambda g(x)$. (4) then becomes

$$\lambda^2 \int_a^b g(x)^2 u^2 \, dx - \lambda \left(u(b)^2 g(b) - u(a)^2 g(a) - \int_a^b g'(x) u^2 \, dx\right) + \int_a^b u'^2 \, dx \geq 0.$$

Since this inequality holds for every real λ, the discriminant of the above quadratic polynomial in λ must be nonpositive, i.e.

$$\left(\int_a^b g' u^2 \, dx + u(a)^2 g(a) - u(b)^2 g(b)\right)^2 \leq 4 \int_a^b g^2 u^2 \, dx \int_a^b u'^2 \, dx.$$

Putting in the last inequality $g(x) = x$, $a = 0$ and letting b tend to infinity, we get WEYL's inequality:

$$\left(\int_0^{+\infty} u^2 \, dx\right)^2 \leq 4 \int_0^{+\infty} x^2 u^2 \, dx \int_0^{+\infty} u'^2 \, dx,$$

provided that the above integrals exist.

D. C. BENSON has cited a number of inequalities which are special cases of Theorems 1 and 2. Some of them are listed below:

1° Let u be a differentiable periodic function of period L and let $m = \inf u(x)$, $M = \sup u(x)$. Then

$$\int_0^L \left(u'^2 + (M - u)^{2\beta} (m - u)^{2\alpha}\right) dx \geq 4 (M - m)^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1)\, \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)},$$

where $\alpha > -\frac{1}{2}$ and $\beta > -\frac{1}{2}$.

2° Let u be a differentiable periodic function of period L. Let $m = \inf u(x)$, $M = \sup u(x)$. Then

$$\int_0^L \left(u'^{2n} + (2n - 1)(u - m)^n (M - u)^n\right) dx \geq \frac{\pi n (2n)!}{(n!)^2\, 2^{4n-2}} (M - m)^{2n},$$

where n is a natural number.

Remark 1. The author — D. C. BENSON — has communicated to us in a letter that the condition for the equality in this inequality, stated in his paper, is not correct. The same holds for some other inequalities established by D. C. BENSON and concerning conditions for the equality. See also the comments in Zentralblatt für Mathematik **146**, 75—76 (1968) by P. R. BEESACK, related to the paper of D. C. BENSON.

Generalization. Let u be an n-times continuously differentiable function, $P(u^{(n-1)}, \ldots, u, x)$ and $G(u^{(n-1)}, \ldots, u, x)$ be continuously differentiable functions and let $P > 0$. Then

$$\int_a^b \Bigl(P(u^{(n)})^{2m} + (2m-1)\, P^{\frac{-1}{2m-1}} G_{u^{(n-1)}}^{\frac{2m}{2m-1}} + 2mu^{(n-1)} G_{u^{(n-2)}} + \cdots + 2mG_x \Bigr)\, dx$$
$$\geq 2m \bigl(G(u^{(n-1)}(b), \ldots, u(b), b) - G(u^{(n-1)}(a), \ldots, u(a), a) \bigr).$$

Equality holds if and only if $u^{(n)} = \left(\frac{1}{P} G_{u^{(n-1)}}\right)^{\frac{1}{2m-1}}$.

The proof of the above inequality is similar to the proof of Theorem 1. The only difference lies in the fact that x in (2) is replaced by

$$u^{(n)} \left(\frac{P}{G_{u^{(n-1)}}}\right)^{\frac{1}{2m-1}}.$$

Remark 2. It would be desirable if the elegant BENSON method were further developed, since the very first applications indicate its fruitfulness and give nice results.

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2.20 Recurrent Inequalities of Redheffer

Definition 1. *Let D_k $(k = 1, \ldots, n)$ be given sets and let*

$$f_k = f_k(a_1, \ldots, a_k) \quad \textit{and} \quad g_k = g_k(a_1, \ldots, a_k)$$

be real-valued functions defined for $a_1 \in D_1, \ldots, a_n \in D_n$.

Inequality

$$\mu_1 f_1 + \cdots + \mu_n f_n \leq g_1 + \cdots + g_n,$$

where μ_k are real parameters, will be called recurrent if and only if there exist functions $F_k(\mu)$ such that

$$\sup_{a_k \in D_k} (\mu f_k - g_k) = F_k(\mu)\, f_{k-1} \qquad (k = 1, \ldots, n;\, f_0 = 1).$$

This definition is due to R. REDHEFFER. For recurrent inequalities R. REDHEFFER [1] has proved the following theorem:

Theorem 1. *The recurrent inequality of the above definition holds for all* $a_k \in D_k$ *if and only if there exists a sequence of real numbers* δ_k *such that* $\delta_1 \leq 0$, $\delta_{n+1} = 0$, *and*

$$\mu_k = F_k^{-1}(\delta_k) - \delta_{k+1} \qquad (k = 1, \ldots, n).$$

$F_k^{-1}(\delta)$ *denotes one of the solutions of the equation* $F_k(\mu) = \delta$.

Proof. We shall prove the above theorem by induction on n. Clearly the theorem holds for $n = 1$. Suppose that it is true for $n - 1$. The inequality will hold for n if and only if it holds for the unfavourable choices of a_n, assuming that the other variables are constant. Since the inequality is recurrent, using the induction hypothesis, we get the relations of the theorem for $k \leq n - 2$ together with

$$F_n(\mu_n) + \mu_{n-1} = F_{n-1}^{-1}(\delta_{n-1}).$$

Defining $\delta_n = F_n(\mu_n)$ we see that the theorem is valid for n, which completes the proof.

Using Theorem 1, various generalizations of known inequalities can be obtained. We illustrate this by an example.

Start from the following inequality

$$\mu_1 G_1 + \cdots + \mu_n G_n \leq \lambda_1 a_1 + \cdots + \lambda_n a_n, \tag{1}$$

where $\lambda_k > 0$ are given numbers, G_k is the geometric mean of $a_1, \ldots, a_k$ and μ_k are parameters which will be determined so as to make the inequality (1) hold for all $a_k > 0$. Inequality (1) is recurrent. To see this, start from the identity

$$\mu G_k - \lambda a_k = G_{k-1}(\mu t - \lambda t^k),$$

where $t^k = \frac{a_k}{G_{k-1}}$, $k > 1$ and $\lambda = \lambda_k$. We therefore get

$$\begin{aligned} F_k(\mu) &= (k-1)\,\lambda^{\frac{1}{1-k}} \left(\frac{\mu}{k}\right)^{\frac{k}{k-1}} && (\mu \geq 0), \\ &= 0 && (\mu < 0). \end{aligned}$$

Therefore $F_1(\mu) = 0$ for $\mu \leq \lambda_1$ and $F_1(\mu) = +\infty$ for $\mu > \lambda_1$. Since $F_k(\mu) \geq 0$ implies $\delta_k \geq 0$, put

$$\delta_k = (k-1)\,\beta_k^{\frac{1}{k-1}}.$$

We then get

$$\mu_k = k\left((\lambda_k \beta_k)^{1/k} - \beta_{k+1}^{1/k}\right),$$

where β_k are arbitrary real numbers such that $\beta_1 \leq 1$, $\beta_k \geq 0$, $\beta_{n+1} = 0$.

Putting $\beta_k = e^{t(k-1)}$ $(t > 0)$, by LAGRANGE's theorem we have

$$k(\beta_k^{1/k} - \beta_{k+1}^{1/k}) > -te^t.$$

Supposing that not all a_k are zero and putting $\lambda_k = 1$, we get

$$G_n < A_n e^{-t} + t\frac{G_1 + \cdots + G_n}{n},$$

where A_n is the arithmetic and G_n the geometric mean of the numbers a_k $(k = 1, \ldots, n)$.

For $t = \frac{G_n}{G} - 1$, where $G = \frac{1}{n}(G_1 + \cdots + G_n)$, we get the following inequality:

$$e(a_1 + \cdots + a_n) \geq e^{G_n/G} \cdot nG.$$

Since $e^{G_n/G} > 1$, we obtain T. CARLEMAN's inequality [2]:

$$G_1 + \cdots + G_n < e(a_1 + \cdots + a_n).$$

Similarly, the following inequality can be obtained

$$\sum_{k=1}^{n} A_k^{1/p} + \frac{n}{1-p} A_n^{1/p} < (1-p)^{-1/p} \sum_{k=1}^{n} a_k^{1/p},$$

where A_k is the arithmetic mean of the numbers $a_1, \ldots, a_k$. This inequality is a generalization of HARDY's inequality [3].

From the above theorem the following inequality can also be deduced

$$a(a_1^2 + \cdots + a_n^2) \leq a_1^2 + (a_2 - a_1)^2 + \cdots + (a_n - a_{n-1})^2 + ba_n^2,$$

which is valid if and only if there is a number $\theta\left(0 \leq \theta < \frac{\pi}{n}\right)$, such that

$$a \leq 2(1 - \cos\theta) \quad \text{and} \quad b \geq 1 - \frac{\sin(n+1)\theta}{\sin n\theta}.$$

In the special case when $\theta = \frac{\pi}{n+1}$ we get the inequality proved by K. FAN, O. TAUSSKY and J. TODD [4]. For $\theta = \frac{\pi}{2n+1}$, another inequality proved by the same authors can also be deduced.

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2.21 Cyclic Inequalities

This Section is concerned with the cyclic inequality

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geq\frac{n}{2}, \tag{1}$$

where

$$x_i\geq 0,\ x_i+x_{i+1}>0 \qquad (x_{n+1}=x_1,\ i=1,\ldots,n), \tag{2}$$

and its generalizations and variations.

The challenge of inequality (1) has recently evoked lively interest among mathematicians, as may be seen from the historical sketch which follows.

As will be seen, at the present time, the only undecided cases of (1) are $n = 11, 12, 13, 15, 17, 19, 21, 23, 25$.

The simplest case for $n = 3$ of this inequality appeared in the literature in 1903 [1] though it is possible that it is not its first appearance.

In 1954 H. S. Shapiro [2] raised the question of proving (1) for all $n = 3, 4, \ldots$ In 1956 a partial answer [3] to this question was obtained. Namely, the editors of the American Mathematical Monthly noted that M. J. Lighthill succeeded in proving that (1) is not true for $n = 20$. The editors also announced that H. S. Shapiro submitted the proof of (1) for $n = 3$ and 4, and C. R. Phelps for $n = 5$. The counterexample of M. J. Lighthill was published in detail in [4].

In 1958 L. J. Mordell [5] has proved that (1) is true for $n = 3, 4, 5, 6$. There is a short note [6] by A. Zulauf appended to this paper of Mordell. In this note he gave a counterexample by which he proved that (1) does not hold for $n = 14$. This result implies that (1) is not true for even $n \geq 14$. Indeed, if we put

$$f_n(x_1,\ldots,x_n)=\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2},$$

then the following functional identity holds

$$f_{n+2}(x_1,\ldots,x_{n-1},x_n,x_{n-1},x_n)=f_n(x_1,\ldots,x_n)+1. \tag{3}$$

Hence, if (1) is false for some n, then it is also false for $n + 2$.

Another counterexample for $n = 14$ was given later, in 1960, by M. Herschorn and J. E. L. Peck [7].

All these counterexamples are of the same kind. Namely, if we set $n = 2m$, $x_{2k} = a_k e$, $x_{2k-1} = 1 + b_k e$ $(k = 1, \ldots, m)$, and e is sufficiently small and positive, then

$$f_{2m}(x_1,\ldots,x_{2m})=m+qe^2+O(e^3) \qquad (e\to 0),$$

where q is a quadratic form in a_k and b_k. If $m = 7$ then $a_k \geq 0$ and b_k can be found such that $q < 0$. For instance, in the counterexample by

ZULAUF we have

$$(a_1, \ldots, a_7) = (7, 6, 5, 2, 0, 1, 4),$$

$$(b_1, \ldots, b_7) = (7, 4, 1, 0, 1, 4, 6) \text{ and } q = -2.$$

DINA GLADYS S. THOMAS [8] has proved that, for $n = 8, 10, 12$, q is a positive definite quadratic form in a_k and b_k. Hence, in these cases there does not exist a counterexample of this kind.

Let us define

$$\mu(n) = \inf_{x_i \geq 0} f_n(x_1, \ldots, x_n), \quad \lambda(n) = \frac{\mu(n)}{n}.$$

Identity (3) implies that

$$\mu(n+2) \leq \mu(n) + 1. \tag{4}$$

In 1958 R. A. RANKIN [9] has proved that there exists $\lambda = \lim_{n \to +\infty} \lambda(n)$ and

$$\lim_{n \to +\infty} \lambda(n) = \inf_{n \geq 1} \lambda(n). \tag{5}$$

He also proved that $\lambda < 1/2 - 7 \cdot 10^{-8}$, which implies that (1) is false for sufficiently large n. In his later paper [10] he proved that $\lambda > 0.33$.

The upper bound for λ was improved by A. ZULAUF [11]. He proved that $\lambda < 0.49950317$. In the same note he proved that (1) does not hold for $n = 53$, i.e., $\lambda(53) < 1/2$. This and (4) imply that $\lambda(n) < 1/2$ for odd $n \geq 53$.

L. J. MORDELL [12] has proved that (1) holds for $n = 7$ if besides (2) we have

$$x_1 \geq x_7 \geq x_2 \geq x_6, \; x_4 \geq x_2, \; x_1 \geq x_3. \tag{6}$$

K. GOLDBERG informed L. J. MORDELL in his letter of February 9, 1960, that he has checked (1) for $n = 7$ and he found it to be true for 300000 pseudo-random values of x_k.

A. ZULAUF [13] remarked that the supplementary inequalities (6) are very restrictive. He proved that if n positive random numbers are chosen, then the probability that (1) is satisfied is at least 1/2. This is implied by the inequality

$$f_n(x_1, \ldots, x_n) + f_n(x_n, x_{n-1}, \ldots, x_1) \geq n.$$

The proof of the last inequality follows: If $A_k = x_k + x_{k+1}$ $(x_{k+n} = x_k)$, then

$$\begin{aligned} f_n(x_1, \ldots, x_n) &+ f_n(x_n, x_{n-1}, \ldots, x_1) \\ &= \sum_{k=1}^{n} \frac{x_k + x_{k+3}}{A_{k+1}} = \sum_{k=1}^{n} \frac{A_k - A_{k+1} + A_{k+2}}{A_{k+1}} \\ &= -n + \sum_{k=1}^{n} \frac{A_k}{A_{k+1}} + \sum_{k=1}^{n} \frac{A_{k+2}}{A_{k+1}} \geq n, \end{aligned}$$

since

$$\sum_{k=1}^{n} \frac{A_k}{A_{k+1}} \geq n, \quad \sum_{k=1}^{n} \frac{A_{k+2}}{A_{k+1}} \geq n.$$

P. H. DIANANDA [14] has obtained the following result: If (2) is satisfied and $x_{m+1}, \ldots, x_{m+n}$ $(x_{k+n} = x_k)$ is monotone for some natural number m, then (1) is true.

The bounds for λ were improved by P. H. DIANANDA in [15] and [16]. Namely, he proved that

$$0.461238 < \lambda < 0.499197.$$

In 1963 D. Ž. ĐOKOVIĆ [17] proved that (1) is true for $n = 8$. P. H. DIANANDA [18] and B. BAJŠANSKI [19] have proved independently of each other that this result of ĐOKOVIĆ implies that (1) is true for $n = 7$. More generally, they proved that

$$\mu(2m) \leq \mu(2m-1) + \frac{1}{2}. \tag{7}$$

P. NOWOSAD [20] has proved that (1) holds for $n = 10$. Therefore, according to (7), inequality (1) holds also for $n = 9$.

P. H. DIANANDA [18] has also proved that (1) does not hold for $n = 27$. The only undecided cases therefore are $n = 11, 12, 13, 15, 17, 19, 21, 23, 25$.

A. ZULAUF [21] has proved that for the modified cyclic sum

$$\sum x_k/(x_k + x_{k+1})$$

the following inequalities hold

$$1 < \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-1} + x_n} + \frac{x_n}{x_n + x_1} < n - 1,$$

where $x_1, \ldots, x_n$ $(n \geq 3)$ are nonnegative, and all denominators positive. These bounds are the best possible.

Let $x_1, \ldots, x_7$ be any nonnegative numbers such that $A_k = x_k + x_{k+1}$ $(x_{k+7} = x_k)$ is positive for $k = 1, \ldots, 7$. In this case A. ZULAUF [22] has proved the following inequalities

$$\sum \frac{x_k}{A_{k+2}} \geq 3, \quad \sum \frac{x_k}{A_{k+4}} \geq 3, \quad \sum \frac{x_k}{A_{k+3}} \geq 2 \tag{8}$$

and some others. All bounds in (8) are the best possible. For instance, putting $(x_1, \ldots, x_7) = (2t^2, t, 0, t^2, t^2, 0, t)$, we obtain

$$\sum \frac{x_k}{A_{k+3}} = 1 + \frac{2}{t} + \frac{2t}{2t+1} \to 2 \qquad (t \to +\infty).$$

In [5] L. J. MORDELL conjectured that

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geq k \sum_{i=1}^{n} x_i (x_{i+1} + \cdots + x_{i+m}), \tag{9}$$

where $x_{n+l} = x_l$, and $x_1, \ldots, x_n$ are all nonnegative, $n \geq m + 1$ and

$$k = \min\left(\frac{n}{m}, \frac{2m+2}{m}\right).$$

He proved this assertion for $n = m + 1$, $n = m + 2$ and for $n \geq 2m$.

It was remarked by T. MURPHY that this conjecture does not hold for $m = 4$, $n = 7$. The best possible value of k is $12/7$, which is smaller than $7/4$.

In the same paper L. J. MORDELL proved that the minimum values F_n, G_n of

$$F_n(x) = \sum_{i=1}^{n} (x_i^2 - 2x_i x_{i+1}) \quad (x_{n+1} = x_1),$$

$$G_n(x) = \sum_{i=1}^{n} x_i^2 - 2 \sum_{i=1}^{n-1} x_i x_{\;+}$$

are

$$F_n = -\frac{1}{n} \quad (n = 3, 4, 5), \qquad F_n = -\frac{1}{6} \quad (n \geq 6),$$

$$G_2 = 0, \quad G_3 = -\frac{1}{7}, \quad G_n = -\frac{1}{6} \quad (n \geq 4),$$

where $x_i \geq 0$ $(i = 1, \ldots, n)$ and $x_1 + \cdots + x_n = 1$.

In [23] P. H. DIANANDA obtained interesting results concerning (9), namely: Let $k(m, n)$ be the largest value of k, such that (9) holds. Then

1° $k(m, n) = \dfrac{n}{m}$ if $n \mid m + 2$ or $2m$ or $2m + 1$ or $2m + 2$,

or if $n \mid m + 3$ and $n = 8$ or 9 or 12,

or if $n \mid m + 4$ and $n = 12$,

$k(m, n) < \dfrac{n}{m}$ otherwise;

2° $k(m, n) = \dfrac{2m+2}{n}$ if $n > 2m + 2$;

3° $k(m, n) = \dfrac{12n}{n + 12m - 6}$ if $n \mid 2m - 1$ and $n > 6$;

4° $k(m, n) = \dfrac{k(m - rn, n)}{1 + rk(m - rn, n)}$ if $rn < m$ $(r = 1, 2, \ldots)$;

5° $k(m, n + 1) \geq k(m, n) \geq k(m + 1, n)$.

V. J. D. BASTON [24] has complemented these results by the following:

6° $\quad k(m, 2m-t) = \dfrac{4(t+2)}{3t+4}$ if $t \neq 3$ and $m \geq \max\left(2t, \dfrac{3t}{2}+2\right)$;

7° $\quad k(7, 11) \leq k(m, 2m-3) \leq \dfrac{20}{13}$ for $m \geq 7$;

8° $\quad k(m, n) \leq \dfrac{2(r+1)\,[(r+1)(m+1) - rn]}{(2r+1)\,[(r+1)\,m - rn] + 2r(r+1)}$ if $m + 2 < n \leq 2m - 1$ and r is an integer such that

$$\frac{r+2}{r+1} \leq \frac{n}{m} < \frac{r+1}{r}.$$

P. H. DIANANDA [25] also considered the following two inequalities

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + \cdots + x_{i+m}} \geq \frac{n}{m}, \tag{10}$$

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geq \frac{n}{m} \sum_{i=1}^{n} x_i (x_{i+1} + \cdots + x_{i+m}), \tag{11}$$

where $x_{n+k} = x_k$ and $x_i > 0$ $(i = 1, \ldots, n)$, and proved the following results:

1° Inequality (10) is true if

$$\sin \frac{r}{n} \pi \geq \sin (2m+1) \frac{r}{n} \pi \qquad \left(r = 1, \ldots, \left[\frac{n}{2}\right]\right). \tag{12}$$

2° Inequality (10) is true if

$$n \mid m+2 \ \text{ or } \ 2m \ \text{ or } \ 2m+1 \ \text{ or } \ 2m+2. \tag{13}$$

3° If (12) is true, then inequality (11) holds.

4° If (13) is true, (12) is true.

In [26] P. H. DIANANDA obtained more general results, as for example,

$$\inf_{x,y,z>0} \left(\frac{x^2}{yz} + \frac{4y^2}{yz+zx} + \frac{(x+2z)^2}{zx+2xy}\right) = 6.$$

Let x_i $(i = 1, \ldots, n)$ be positive real numbers and $x_{n+k} = x_k$. P. H. DIANANDA in three papers [16], [27], [28] respectively has proved the following results:

$$\inf_{x_1,\ldots,x_n} \frac{4}{n} \sum_{i=1}^{n} \frac{x_i}{3x_{i+1} + x_{i+2} + |x_{i+1} - x_{i+2}|} = 2^s - s \qquad \left(s = \frac{1}{n}\left[\frac{n}{3}\right]\right),$$

$$\inf_{x_1,\ldots,x_n} \sum_{i=1}^{n} \frac{x_i^2}{x_{i+1}^2 - x_{i+1}x_{i+2} + x_{i+2}^2} = \left[\frac{n+1}{2}\right],$$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + \cdots + x_{i+m}} \geq \frac{1}{m}\left[\frac{n+m-1}{m}\right] \geq \frac{n}{m^2}.$$

Finally, we note the following cyclic inequality [29]

$$\frac{x_1 + \cdots + x_k}{x_{k+1} + \cdots + x_n} + \frac{x_2 + \cdots + x_{k+1}}{x_{k+2} + \cdots + x_1} + \cdots + \frac{x_n + x_1 + \cdots + x_{k-1}}{x_k + \cdots + x_{n-1}} \geq \frac{nk}{n-k},$$

where $x_1, \ldots, x_n$ are positive numbers, and $n > k \geq 1$.

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2.22 Inequalities Involving Derivatives

Let f be a real function, defined and bounded, together with its first n derivatives on R, and let for $k = 0, 1, \ldots, n$

$$M_k = \sup |f^{(k)}(x)| \quad (x \in R).$$

G. H. Hardy and J. E. Littlewood initiated in 1914 the problem of determining the constant $C_{n,k}$ for general n in the inequality

$$M_k^n \leq C_{n,k}^n M_0^{n-k} M_n^k.$$

J. Hadamard proved in 1914 that $C_{n,1} \leq 2^{n-1}$ and, as a special case, obtained the inequality

$$M_1^2 \leq 2M_0 M_2. \tag{1}$$

The result $C_{2,1} = \sqrt{2}$ also appears in a paper of E. Landau from 1913.

Concerning the references of the problem in question, see an interesting review of paper [7] by R. P. Boas in Math. Reviews **1**, 298 (1940).

G. E. Šilov [1] solved the problem for some k and n. In fact, he proved the following inequalities:

$$M_1^2 \leq 2M_0 M_2 \qquad \text{for } k = 1 \text{ and } n = 2;$$

$$M_1^3 \leq \frac{9}{8} M_0^2 M_3 \qquad \text{for } k = 1 \text{ and } n = 3;$$

$$M_2^3 \leq 3M_0 M_3^2 \qquad \text{for } k = 2 \text{ and } n = 3;$$

$$M_1^4 \leq \frac{512}{375} M_0^3 M_4 \qquad \text{for } k = 1 \text{ and } n = 4;$$

$$M_2^2 \leq \frac{6}{5} M_0 M_4 \qquad \text{for } k = 2 \text{ and } n = 4;$$

$$M_2^5 \leq \frac{125}{72} M_0^3 M_5^2 \qquad \text{for } k = 2 \text{ and } n = 5,$$

the first of which is Hadamard's inequality (1). These inequalities are the best possible. See also papers of A. M. Rodov, reviewed in Math. Reviews **8**, 65 (1947) and **17**, 716 (1956).

A. Gorny, unifying his results published in [2], [3] and [4], but in a somewhat modified form, proved in [5] the following result:

If f is an n-times differentiable function on a closed interval I of length δ, and if $|f(x)| \leq M_0$ and $|f^{(n)}(x)| \leq M_n$, then for $x \in I$ and for $0 < k < n$,

$$|f^{(k)}(x)| \leq 4e^{2k} \binom{n}{k}^k M_0^{1\quad(k/n)} M_n'^{k/n},$$

while, at the midpoint of I,

$$|f^{(k)}(x)| \leq 16(2e)^k M_0^{1-(k/n)} M_n'^{k/n},$$

where

$$M_n' = \max(M_n, M_0\, n!\, \delta^{-n}).$$

If $I = (0, +\infty)$ or $(-\infty, +\infty)$, then $M_n' = M_n$ and for $(-\infty, +\infty)$ any point can be considered as the midpoint. He thus proved that

$$C_{n,k} \leq 16(2e)^k.$$

The complete solution of the quoted problem was given by A. Kolmogoroff in [6] and [7] who proved that

$$C_{n,k} = K_{n-k}/K_n^{(n-k)/n},$$

where

$$K_n = \frac{4}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \quad \text{for } n \text{ even},$$

and

$$K_n = \frac{4}{\pi} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^{n+1}} \quad \text{for } n \text{ odd}.$$

He also showed that $1 < C_{n,k} < \frac{\pi}{2}$ for all n and k, and that $C_{n,n-1} \to \frac{\pi}{2}$ and $C_{n,1} \to 1$ both as $n \to +\infty$.

The same problem, determining $C_{n,k}$ in (1), was treated by S. B. Stečkin [8] for the interval $(0, +\infty)$. He proved that

$$a(n/k)^k \leq C_{n,k} \leq A\left(e^2 n/(4k)\right)^k, \qquad 1 \leq k \leq n/2,$$

$$a(n-k)^{-1/2}\left(n/(n-k)\right)^{n-k} \leq C_{n,k} \leq A\left(2n/(n-k)\right)^{n-k}, \quad n/2 \leq k < n,$$

where a and A are positive constants.

These inequalities present improvements and simplifications of inequalities obtained by V. M. Olovyanišnikov and A. P. Matorin which are discussed in [8].

Just recently, V. A. Dubovik and B. I. Korenbljum, in Mat. Zametki **5**, 13—20 (1969), considered the problem of Hadamard and Kolmogoroff, with a complementary condition on the derivative $f^{(k)}(x)$.

Besides the ones cited above, there exist numerous inequalities which involve derivatives. We shall quote some of them.

If f is an n-times differentiable function of the variable $t \in [a, b]$ and if $a_1, \ldots, a_n$ $(a \leq a_1 \leq \cdots \leq a_n \leq b)$ are zeros of f, then for $a \leq t \leq b$ and for $k = 0, 1, \ldots, n-1$,

$$|f^{(k)}(t)| \leq \frac{1}{(n-k)!}(b-a)^{n-k} \max_{a \leq t \leq b} |f^{(n)}(t)|.$$

The above estimate is improved in the paper [9] by G. A. Bessmertnyh and A. Yu. Levin.

In the book [10] the following result is given without a proof:

Let f be an n-times differentiable real function defined in $(-1, 1)$ and such that $|f(t)| \leq 1$ in that interval. Let $m_k(J)$ be the smallest value of $|f^{(k)}(t)|$ in an interval J contained in $(-1, 1)$. If J is decomposed into three consecutive intervals J_1, J_2, J_3, and if J_2 has the length μ, then

$$m_k(J) \leq \frac{1}{\mu}\left(m_{k-1}(J_1) + m_{k-1}(J_3)\right).$$

From the above inequality follows

$$m_k(J) \leq \frac{2^{\frac{1}{2}k(k+1)}\,k^k}{\lambda^k},$$

where J has the length λ.

Finally, we mention the following result due to G. Aumann [11].

If a complex function $F(x)$ is in $a \leq x \leq b$ twice continuously differentiable, and if it is not constant in any subinterval; furthermore, if $F(a) = 0$, $F(b) = 2\gamma \geq 0$ and $\mu = \max\limits_{a \leq x \leq b} |\operatorname{Im} F(x)| > 0$, then

$$\min_{a \leq x \leq b} \left|\frac{F'(x)^2}{F''(x)}\right| \leq \max\left(\mu, \frac{\gamma^2 + \mu^2}{2\mu}\right).$$

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2.23 Integral Inequalities Involving Derivatives

2.23.1 An Inequality Ascribed to Wirtinger

Let f be a periodic real function with period 2π and let $f' \in L^2$. Then, if $\int_0^{2\pi} f(x)\,dx = 0$, the following inequality holds

$$\int_0^{2\pi} f(x)^2\,dx \leq \int_0^{2\pi} f'(x)^2\,dx \tag{1}$$

with equality if and only if $f(x) = A \cos x + B \sin x$, where A and B are constants.

Inequality (1) is known in the literature as WIRTINGER's inequality (see [1] and [2]). As far as we know, the proof of W. WIRTINGER was first published in 1916 in the book [3] of W. BLASCHKE.

However, inequality (1) was known before this, and with weaker conditions on the function f. For example, in 1905, E. ALMANSI [4] has proved that

$$\int_a^b f'(x)^2\,dx \geq \left(\frac{2\pi}{b-a}\right)^2 \int_a^b f(x)^2\,dx, \tag{2}$$

under the condition that f and f' are continuous on the interval (a, b), that $f(a) = f(b)$, and that $\int_a^b f(x)\,dx = 0$.

Those conditions were weakened, in 1911, by E. E. LEVI [5] and, again in 1914, by L. TONELLI [6]. However, inequalities of the form (2), as well as more general inequalities, can be found even before ALMANSI's result. For example, in the book [7] of É. PICARD the problem of finding the function f which maximizes the expression

$$\frac{\int_a^b p(x)\, f(x)^2\,dx}{\int_a^b f'(x)^2\,dx}$$

was considered, where f and f' are continuous functions, $f(a) = f(b)$ and where p is a positive continuous function on (a, b). This problem under other conditions was later considered by P. R. BEESACK [8], of which we shall talk later.

Inequality (2) can be also found in the book [9] of J. HADAMARD from 1910.

H. A. SCHWARZ in his paper [10] from 1885 has determined the maximum value of the quotient

$$\frac{\iint_T p(x, y) f(x, y)^2 \, dx \, dy}{\iint_T \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dx \, dy}.$$

H. POINCARÉ [11] proved that the above quotient of SCHWARZ for $p(x, y) \equiv 1$ is less than $7l^2/24$, where T is a convex region and f is a function such that $\iint_T f(x, y) \, dx \, dy = 0$ and where l is the maximal chord of that region.

In the same article, H. POINCARÉ considered the analogous problem for a three-dimensional region and gave the estimate for the corresponding quotient.

E. E. LEVI [12] studied the same quotient for the case $p(x, y) \equiv 1$ under somewhat different conditions and proved the following result:

If T is a convex region with respect to a point O whose minimum and maximum distance from the contour of T is l and L respectively, and if $\iint_T f(x, y) \, dx \, dy = 0$, then this quotient is less than $1/K$ where $K = \min\left(\frac{3l}{2L^3}, \frac{1}{13L^2}\right)$.

E. E. LEVI [13] has proved a number of inequalities involving the same integrals as in (1), together with some other inequalities whose form is analogous to (1). Thus, for example, he proved the following result:

Let f be a function whose derivative is bounded on (a, b), and such that $f(a) = f(b) = 0$. Furthermore, let $|f(x)| < \alpha$. Divide the interval (a, b) into two complementary measurable subsets k_1 and k_2. Then

$$\int_a^b f(x)^2 \, dx \leq \frac{(b-a)^2}{8} \int_{k_1} f'(x)^2 \, dx + \alpha (b - a) \int_{k_2} |f'(x)| \, dx,$$

$$\int_a^b |f(x) \, f'(x)| \, dx \leq \frac{(b-a)^2 + 8}{16} \int_{k_1} f'(x)^2 \, dx + \alpha \left(\frac{b-a}{2} + 1\right) \int_{k_2} |f'(x)| \, dx.$$

We have not succeeded in finding an older source than that from 1885. Actually, our attention was drawn to it by Professor M. JANET in our correspondence concerning the priority of WIRTINGER's inequality.

A. PLEIJEL [14] has proved that

$$2\pi \frac{\int_0^{2\pi} f'(x)^2 \, dx}{\int_0^{2\pi} f''(x)^2 \, dx} \int_0^{2\pi} f'(x)^2 \, dx \leq 2\pi \int_0^{2\pi} f(x)^2 \, dx - \left(\int_0^{2\pi} f(x) \, dx\right)^2$$

$$\leq 2\pi \int_0^{2\pi} f'(x)^2 \, dx,$$

where f is a real function of period 2π, with a second derivative in L^2. The second inequality above is, in fact, an improvement of (2). This inequality was established earlier by M. JANET in [15] in a somewhat more general form.

Inspired by papers [15] and [16] of M. JANET, G. CIMMINO [17] has proved the following result: Let $p < n$, and let the function f together with its first $n - 1$ derivatives be continuous on (a, b) with $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0$. Then

$$\frac{\int_a^b f^{(n)}(x)^2\,dx}{\int_a^b f^{(p)}(x)^2\,dx} \geq \left(\frac{v_{n,p}}{b-a}\right)^{2n-2p}, \tag{3}$$

where $v_{n,p}$ is the least positive zero of the Wronskian of n independent solutions of the equation

$$f^{(2n)}(x) - (-1)^{n+p} f^{(2p)}(x) = 0.$$

Later, M. JANET, in papers [18] and [19] considered the problem of determining the minimum value of (3) explicitly.

The following result due to M. JANET ([18], [19]) is also interesting: Let (C) be the curve described by the point whose Cartesian coordinates are

$$\xi = \frac{t(1+\cos t)}{t+\sin t}, \qquad \eta = t^2\frac{t-\sin t}{t+\sin t} \qquad (0 \leq t \leq \pi),$$

and

$$\xi = \frac{t(1+\cosh t)}{t+\sinh t}, \qquad \eta = t^2\frac{\sinh t - t}{\sinh t + t} \qquad (0 \leq t).$$

This curve consists of two branches: one which decreases from $(0, \pi^2)$ to $(1, 0)$; and the other which increases from $(1, 0)$ to $(+\infty, +\infty)$. The point

$$X = \frac{\frac{1}{2}(f(0)^2 + f(1)^2)}{\int_0^1 f(x)^2\,dx}, \qquad Y = \frac{\int_0^1 f'(x)^2\,dx}{\int_0^1 f(x)^2\,dx},$$

where f and f' are continuous functions on $(0, 1)$, can only belong to the region above C, or to C itself.

For some more results which are related to (3), see the papers [20] – [22] of M. JANET, [23] of G. CIMMINO and the doctoral dissertation [24] of TCHENG TCHOU-YUN.

E. SCHMIDT [25] has proved the following result:

Let z be a continuous function on $[0, \lambda]$, with $z(0) = z(\lambda)$, $\min_{0\leq t\leq\lambda} z(t) = m$, $\max_{0\leq t\leq\lambda} z(t) = M$ and $M + m = 0$. Suppose that its derivative z' is defined and continuous except at most in a finite number of points; furthermore let it be absolutely integrable.

Then

$$\left(\frac{1}{\lambda}\int_0^\lambda |z(t)|^a\,dt\right)^{1/a} \leq \frac{1}{4} H\left(\frac{1}{a},\ \frac{b-1}{b}\right)\left(\lambda^{b-1}\int_0^\lambda |z'(t)|^b\,dt\right)^{1/b},$$

where $a > 0$, $b \geq 1$, and

$$H(u, v) = \frac{u^u v^v}{(u+v)^{u+v}} \cdot \frac{\Gamma(1+u+v)}{\Gamma(1+u)\,\Gamma(1+v)}.$$

If the conditions under which the above inequality holds are altered so that $a = b \geq 1$, and if the condition $m + M = 0$ is omitted, then, for $0 < t < \lambda$, it becomes

$$\int_0^\lambda \left|z(t) - \frac{1}{2}(m+M)\right|^b dt \leq \frac{1}{b-1}\left(\frac{b}{4\pi}\sin\frac{\pi}{b}\right)^b \lambda^b \int_0^\lambda |z'(t)|^b\,dt,$$

from where for $b = 2$, together with the supplementary condition $\int_0^\lambda z(t)\,dt = 0$, we obtain an improvement of WIRTINGER's inequality

$$\int_0^\lambda z(t)^2\,dt \leq \frac{\lambda^2}{4\pi^2}\int_0^\lambda z'(t)^2\,dt - \lambda\left(\frac{m+M}{2}\right)^2.$$

For this inequality in the case $\lambda = 2\pi$, see 2.19.

In connection with this result of E. SCHMIDT see also the papers [26] of R. BELLMAN and [27] of B. SZ.-NAGY.

Generalizing a result of D. G. NORTHCOTT [28], R. BELLMAN [29] has proved the inequality

$$\int_{-\pi}^{+\pi} f(x)^{2k}\,dx \leq a_n^{2k}\int_{-\pi}^{+\pi} f^{(n)}(x)^{2k}\,dx, \tag{4}$$

where k, n are natural numbers and a_n are constants (e.g., $a_1 = \pi/2$, $a_2 = \pi^2/8$, etc.) under the condition that $f^{(n)} \in L^{2k}$, $f(x) = f(x + 2\pi)$ and $\int_{-\pi}^{+\pi} f(x)\,dx = 0$.

Notice that, for $k = 1$ and $n = 1$, inequality (4) is weaker than (1).

In [8], P. R. BEESACK has given several generalizations of inequality (1). We quote the one which was mentioned earlier:

Suppose that the differential equation

$$y''(x) + p(x)\,y(x) = 0, \tag{5}$$

where p is a continuous function on some interval $(-a, a)$, with $a > 0$, has a solution $y_1(x) > 0$ for $x \in (-a, a)$, and that $\int_{-a}^{+a} p(x)\,dx \geq 0$. Then, if $f(-a) = f(a)$, $f' \in L^2$ and $\int_{-a}^{+a} p(x)\, f(x)\, dx = 0$, the following inequality holds:

$$\int_{-a}^{+a} f'(x)^2\, dx \geq \int_{-a}^{+a} p(x)\, f(x)^2\, dx.$$

Equality holds here if and only if $f(x) = Ay_1(x)$, where $A = 0$ if either $y_1(-a) \neq 0$ or $y_1(a) \neq 0$.

In the same paper, P. R. BEESACK has also considered inequalities of the form

$$\int_{-a}^{+a} f''(x)^2\, dx \geq \int_{-a}^{+a} p(x)\, f(x)^2\, dx.$$

We shall also mention the following inequality of W. J. KIM [30]:

Let f be a continuous function on $[a, b]$ together with its derivatives up to m-th order, and let it, together with its first $m - 1$ derivatives, be equal to zero for $x = a$ and $x = b$. Then

$$\int_a^b f^{(m)}(x)^2\, dx > \left(\frac{b-a}{2}\right)^{2m} \prod_{k=0}^{m-1} (2k+1)^2 \int_a^b \frac{f(x)^2}{(a-x)^{2m}\,(b-x)^{2m}}\, dx.$$

This result is in connection with the results obtained by P. R. BEESACK [8], and Z. NEHARI [31].

P. R. BEESACK [32] has also proved the following generalization of inequality (1): Let $f' \in L^{2k}$ on $[-\pi, +\pi]$, $f(-\pi) = f(\pi)$ and $\int_{-\pi}^{+\pi} f(x)^{2k-1}\, dx = 0$. Then the following inequality holds

$$\int_{-\pi}^{+\pi} f(x)^{2k}\, dx \leq \frac{1}{2k-1}\left(k \sin\frac{\pi}{2k}\right)^{2k} \int_{-\pi}^{+\pi} f'(x)^{2k}\, dx. \tag{6}$$

In this paper, cases of equality were also considered.

In the same article P. R. BEESACK has proved the following result: Let r, r', s be continuous functions on (a, b) and let $r(x) > 0$, $s(x) \geq 0$ on that interval, except that r may have a single zero or a single discontinuity at a point x $(a < x < b)$. We shall say that the function f is an integral on (a, b) if for all $c \in (a, b)$,

$$f(x) = f(c) + \int_c^x f'(t)\, dt \qquad (a < x < b).$$

Let $p = 2k$, $q = \dfrac{2k}{2k-1}$ and let the differential equation

$$\frac{d}{dx}\left(r(x)\left(\frac{dy}{dx}\right)^{p-1}\right) + s(x)\, y^{p-1} = 0$$

have a solution $y(x)$ which is an integral on (a, b) and $y(x) < 0$ for $a < x < \bar{x}$ and $y(x) > 0$ for $\bar{x} < x < b$. Suppose that

$$\frac{y'(x)}{y(x)} = O\left(\frac{1}{x - \bar{x}}\right), \quad \frac{y'(x)}{y(x)} = O\left(\frac{1}{x - a}\right), \quad \frac{y'(x)}{y(x)} = O\left(\frac{1}{b - x}\right)$$

for x near $\bar{x}$, a, b respectively, and that r satisfies the three conditions

$$r(x) = O\left((x - \bar{x})^{p-1}\right) \quad \text{or} \quad r(x)^{q/p} \int_{\bar{x}}^{x} r(t)^{-q/p}\, dt = O(x - \bar{x}),$$

$$r(x) = O\left((x - a)^{p-1}\right) \quad \text{and} \quad r(x)^{q/p} \int_{\bar{x}}^{k_1} r(t)^{-q/p} dt = O(x - a),$$

$$r(x) = O\left((b - x)^{p-1}\right) \quad \text{and} \quad r(x)^{q/p} \int_{k_2}^{x} r(t)^{-q/p}\, dt = O(b - x),$$

for x near $\bar{x}$, a, b respectively. Here k_1, k_2 denote any constants such that $k_1 \le \bar{x} \le k_2$.

Now, let f be an integral on (a, b), and suppose that

$$\int_a^b r(x)\, f'(x)^p\, dx < +\infty, \quad \int_a^b s(x)\, f(x)^{p-1}\, dx = 0.$$

Then

$$\int_a^b s(x)\, f(x)^p\, dx \le \int_a^b r(x)\, f'(x)^p\, dx$$

with equality if and only if $f(x) = c y(x)$, where $c = 0$ unless

$$\int_a^b r(x)\, y'(x)^p\, dx < +\infty.$$

Further generalizations of inequality (1) were given by W. J. COLES in [33] and [34]. One of them reads:

Let m be a natural number, $n = 2m$ and let k_i $(0 \le i \le m)$ be either 0 or 1 so that $\sum_{i=0}^{m} k_i$ is an even number. Let $h_i = \sum_{j=0}^{i} k_j$, $p_i = (-1)^{h_i}$, $q_i = (-1)^i p_i$. Let p be a real continuous function on $[a, b]$, $c_i = (1 - k_i)\, a + k_i b$, $d_i = k_{i+1} a + (1 - k_{i+1})\, b$ and $d'_i = a + b - d_i$. If the differential equation $y^{(n)}(x) - p(x)\, y(x) = 0$ has a solution y such that $(-1)^m p(x)\, y(x) \ge 0$ (but not identically equal to zero) on $[a, b]$, $p_i y^{(m-i)}(c_i) \ge 0$ $(1 \le i \le m)$ and $q_i y^{(m+i)}(d_i) \ge 0$ $(0 \le i \le m - 1)$, then the following inequality holds for each function f such that $f^{(m)}(x) \in L^2$ and $f^{(i)}(d'_{m-i-1}) = 0$ of at least order 1 $(0 \le i \le m - 1)$:

$$(-1)^m \int_a^b p(x)\, f(x)^2\, dx \le \int_a^b f^{(m)}(x)^2\, dx.$$

The case of equality was also discussed, but the conditions are too complicated to be included here.

W. J. Coles has also considered inequalities of the form

$$\sum_{i=0}^{n}(-1)^{n+i}\int_a^b p_i(x)\, f^{(i)}(x)^2\, dx \geq 0,$$

where p_i and f are functions which satisfy conditions which we shall not quote here.

J. B. Diaz and F. T. Metcalf [35] have proved a number of inequalities which generalize inequality (2). Their main results are inequalities (7), (8) and (9) below.

1° Let the real-valued function f be continuously differentiable on the finite interval $[a, b]$. Let t_1 and t_2 be real numbers such that $a \leq t_1 \leq t_2 \leq b$ and $f(t_1) = f(t_2)$. Then

$$\begin{aligned}(7)\qquad &\int_a^b (f(x) - f(t_1))^2\, dx \\ &\leq \frac{4}{\pi^2}\max\left((t_1 - a)^2,\ (b - t_2)^2,\ \left(\frac{t_2 - t_1}{2}\right)^2\right)\int_a^b f'(x)^2\, dx.\end{aligned}$$

2° If the function f satisfies the same conditions as in 1°, together with $f(a) = f(b)$, then

$$\begin{aligned}(8)\qquad &\int_a^b (f(x) - f(t_1))^2\, dx \\ &\leq \frac{1}{\pi^2}\max\left((t_2 - t_1)^2,\ (b - a - t_2 + t_1)^2\right)\int_a^b f'(x)^2\, dx.\end{aligned}$$

3° If the function f satisfies the same conditions as in 2°, together with

$$(b - a)\, f(t_1)^2 \geq 2f(t_1)\int_a^b f(x)\, dx,$$

then

$$(9)\qquad \int_a^b f(x)^2\, dx \leq \frac{1}{\pi^2}\max\left((t_2 - t_1)^2,\ (b - a - t_2 + t_1)^2\right)\int_a^b f'(x)^2\, dx.$$

In all three inequalities, conditions for equality were discussed, but the results obtained are too complicated to be included here.

Inequalities of the form (2) can be extended in the following way:

If f possesses a continuous second derivative on $[a, b]$, then for every $\varepsilon > 0$,

$$(10)\qquad \int_a^b f'(x)^2\, dx \leq \varepsilon\int_a^b f''(x)^2\, dx + K(\varepsilon)\int_a^b f(x)^2\, dx,$$

where

$$K(\varepsilon) = \frac{P}{\varepsilon} + \frac{Q}{(b-a)^2} \quad \text{and} \quad P = 1,\ Q = 12.$$

The constants $P = 1$ and $Q = 12$ are the best possible in the sense that if $P < 1$ or $Q < 12$, then (10) does not hold for all f and all $\varepsilon > 0$.

For this result see I. Halperin and H. Pitt [36], W. Müller [37], L. Nirenberg [38] and R. Redheffer [39].

Instead of inequality (2), A. M. Pfeffer [40] has considered, among others the following best possible inequality

$$\int_a^b f^{(k)}(x)^2\,dx \le \varepsilon \int_a^b f^{(m)}(x)^2\,dx + H_{k,m}(\varepsilon) \int_a^b f(x)^2\,dx, \tag{11}$$

and has proved that it holds under the condition: $f(x) \in C^m[a, b]$, $1 \le k < m$, $f(a) = f(b)$, $f^{(i)}(a) = f^{(i)}(b)$ $(i = 1, \ldots, m-1)$ and $\varepsilon > 0$.

In (11) $H_{k,m}$ denotes

$$H_{k,m}(\varepsilon) = 0 \quad \text{for} \quad \varepsilon > \left(\frac{b-a}{2\pi}\right)^{2m-2k},$$

$$H_{k,m}(\varepsilon) = \left(\frac{k}{m\varepsilon}\right)^{\frac{k}{m-k}} \left(1 - \frac{k}{m}\right) \quad \text{if} \quad \frac{b-a}{2\pi}\left(\frac{k}{m\varepsilon}\right)^{\frac{1}{2m-2k}} \text{ is a positive integer,}$$

and otherwise

$$H_{k,m}(\varepsilon) = \max\left(\left(\frac{2J\pi}{b-a}\right)^{2k} - \varepsilon\left(\frac{2J\pi}{b-a}\right)^{2m}, \left(\frac{2(J+1)\pi}{b-a}\right)^{2k} - \varepsilon\left(\frac{2(J+1)\pi}{b-a}\right)^{2m}\right),$$

where

$$J = \left[\frac{b-a}{2\pi}\left(\frac{k}{m\varepsilon}\right)^{\frac{1}{2m-2k}}\right].$$

Following an observation of P. R. Beesack, we have replaced in the above formulas the incorrect constant $\left(\frac{k}{m\varepsilon}\right)^{k/m-k}$ by its true value $\left(\frac{k}{m\varepsilon}\right)^{\frac{k}{m-k}}$.

B. A. Troesch [41] has established the following result:

Let h be a positive function with piecewise smooth derivative on $I = [0, 1]$ such that $-h$ is convex on I and $h'(0) \le 0$. Let f be continuous and piecewise smooth on I with $f(0) = 0$. Then

$$\frac{\int_0^1 h(x) f'(x)^2\,dx}{\int_0^1 h(x)\,dx \int_0^1 f(x)^2\,dx} \ge \frac{\pi^2}{4},$$

with equality if and only if h is a constant and $f(x) = A \sin\left(\frac{1}{2}\pi x\right)$, where A is a constant.

In the problem whose author is not cited, and which was proposed and solved in [42], the inequality

$$\int_T f(x_1, \ldots, x_n)\, dT \leq \frac{1}{n}\left(\frac{D}{\pi}\right)^2 \int_T \left[\left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2\right] dT$$

was proved, where f is a finite and a continuous function together with its partial derivatives in the region T which is bounded and measurable, and where f vanishes at the boundary of T. D denotes the diameter of T.

A. WEINSTEIN [43] studied the quotients I/H, I/D, D/H, where

$$H = \iint_T f(x, y)^2\, dx\, dy,$$

$$D = \iint_T \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dx\, dy,$$

$$I = \iint_T \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^2 dx\, dy$$

determining their minimum values. He reduced this problem to finding eigenvalues of corresponding partial differential equations, using the methods of the calculus of variations. In connection with that he only mentions article [20] of M. JANET, though the quotient D/H was studied by H. POINCARÉ [11] and E. E. LEVI [12]. A. WEINSTEIN obtained some interesting results which are too long to be quoted here.

D. M. MANGERON [44] has generalized inequality (2) to functions of several variables, but in a direction which differs from the cited result [10] of H. A. SCHWARZ. In fact, D. M. MANGERON has determined, under certain conditions which we shall not quote here, the minimum value of the quotient

$$\frac{\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} p(x_1, \ldots, x_m) \left(\frac{\partial^m f(x_1, \ldots, x_m)}{\partial x_1 \cdots \partial x_m}\right)^2 dx_1 \cdots dx_m}{\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} q(x_1, \ldots, x_m)\, f(x_1, \ldots, x_m)^2\, dx_1 \cdots dx_m}.$$

H. D. BLOCK [45] has formed a class of integral inequalities which contain inequalities of WIRTINGER's type.

V. I. LEVIN and S. B. STEČKIN [46] have indicated a number of interesting analogues of inequality (1).

D. W. BOYD [47] has given a method for determining the best possible constants K in the inequalities of the form

$$(12) \qquad \int_a^b |f(x)|^p\, |f^{(n)}(x)|^q\, w(x)\, dx \leq K \left(\int_a^b |f^{(n)}(x)|^r\, v(x)\, dx\right)^{\frac{p+q}{r}},$$

where $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, and where $f^{(n-1)}$ is an absolutely continuous function.

If v and w are sufficiently smooth functions, the solution of the problem in question reduces to a boundary value problem for a differential equation. D. W. BOYD illustrated his interesting method by determining the best constants in the case when (a, b) is a finite interval, and $v(x) \equiv w(x) \equiv 1$, $n = 1$.

Before BOYD's result, O. ARAMĂ and D. RIPIANU [48] gave two particular cases of (12):

$$v(x) \equiv w(x) \equiv 1, \quad n = 1, \quad p = q = r = 2,$$

and

$$v(x) \equiv w(x) \equiv 1, \quad n = 1, \quad p = 4, \quad q = 0, \quad r = 2.$$

However, the constants used by O. ARAMĂ and D. RIPIANU are not the best possible, even more so in the second case as there is a complementary condition $f(b) = 0$.

Inequality (12) appears to be the first to connect the WIRTINGER and the OPIAL inequalities. For OPIAL'S inequality, see 2.23.2.

Results on discrete analogues of inequality (1) and its variations which will now be listed are due to K. FAN, O. TAUSSKY and J. TODD [49].

If $x_1, \ldots, x_n$ is a sequence of n real numbers, with $x_1 = 0$, then

$$\sum_{k=1}^{n-1} (x_k - x_{k+1})^2 \geq 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{k=2}^{n} x_k^2$$

with equality if and only if

$$x_k = \sin \frac{(k-1)\pi}{2n-1} \qquad (k = 1, \ldots, n).$$

We also have

$$(13) \quad \sum_{k=1}^{n} (x_k - x_{k+1})^2 \geq 4 \sin^2 \frac{\pi}{n} \sum_{k=1}^{n} x_k^2 \qquad \left(x_{n+1} = x_1;\ \sum_{k=1}^{n} x_k = 0\right),$$

with equality if and only if

$$x_k = A \cos \frac{2k\pi}{n} + B \sin \frac{2k\pi}{n} \qquad (k = 1, \ldots, n).$$

Notice that, passing to the limit in (13), we get (1).

Discrete analogues of inequality (3) and some other inequalities were also given in the mentioned article [49]. In connection with results from [49], see also the paper [50] of H. D. BLOCK.

A. M. PFEFFER [40] has proved the discrete analogues of inequalities (10) and (11).

From the exposition made above it follows that there were two almost independent trends for determining the estimates of quotients as are, for instance,

$$\frac{\int_a^b p(x)\, f(x)^2\, dx}{\int_a^b f'(x)^2\, dx}, \qquad \frac{\iint_T p(x, y)\, f(x, y)^2\, dx\, dy}{\iint_T \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dx\, dy}.$$

In fact, there is also a third orientation in the study of the quotients in question which is closely related to some problems of mathematical physics. So, for example, if

$$\Lambda = \min_f \left(\frac{\iint_T \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dx\, dy}{\iint_T f(x, y)^2\, dx\, dy} \right)^{1/2},$$

then

$$\Lambda < \frac{\pi}{2} \frac{L}{A},$$

where T is a simply connected plane domain, L the length of the perimeter of T, A the area of T, and where f, $\partial f/\partial x$, $\partial f/\partial y$ are continuous in T and $f(x, y) = 0$ on the boundary of T.

The above inequality, which is the best possible, was proved by G. Pólya [51].

E. Makai [52] has proved that if T is a convex domain, then

$$\Lambda \geq \frac{\pi}{4} \frac{L}{A},$$

and this estimate is the best possible.

On these topics see in particular the book [53] of G. Pólya and G. Szegö.

Papers [52] and [53] involve the comprehensive literature related to the considered subject.

Remark 1. In papers [54] and [55] the history of inequality (1) as well as of some inequalities connected with (1) was given. The attention was also drawn to the fact that the name Wirtinger's inequality is not justified for (1).

Remark 2. For some generalizations of results in [49] see a paper of R. H. Sadikova published in Volž. Mat. Sb. 1969, No. 7, 161–165.

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2.23.2 An Inequality of Opial

Z. Opial [1] has proved in 1960 the following result:

Theorem 1. *If f' is a continuous function on $[0, h]$, and if $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0, h)$, then*

$$\int_0^h |f(x)\, f'(x)|\, dx \leq \frac{h}{4} \int_0^h f'(x)^2\, dx, \tag{1}$$

where the constant $h/4$ is the best possible.

This result of Z. Opial has led to numerous articles whose contents can be classified into three groups: 1° those whose aim is to give the simplest proof of inequality (1); 2° those which give various generalizations of inequality (1); and 3° those which aim to find discrete analogues of Opial's inequality and its generalizations.

The first result to follow Z. Opial is due to C. Olech [2]. He has weakened the conditions of Theorem 1 and proved

Theorem 2. *Let f be an absolutely continuous function on $[0, h]$ and let $f(0) = f(h) = 0$. Then inequality* (1) *holds.*

Equality holds if and only if

$$f(x) = cx \quad \text{for} \quad 0 \leq x \leq h/2 \quad \text{and} \quad f(x) = c(h - x) \quad \text{for} \quad h/2 \leq x \leq h,$$

where c is a constant.

Olech's proof is simpler than Opial's.

In order to prove (1), it is sufficient to prove the following:

Theorem 2'. *If g is an absolutely continuous function on $[0, a]$ and if $g(0) = 0$, the following inequality holds*

$$\int_0^a |g(x)\, g'(x)|\, dx \leq \frac{a}{2} \int_0^a g'(x)^2\, dx, \tag{2}$$

where $a/2$ is the best possible constant.

Equality in (2) *holds if and only if $f(x) = cx$, where c is a constant.*

Indeed, putting in (2) $f = g$, $a = h/2$ and then $g(x) = f(h - x)$, $a = h/2$, we get the following inequalities:

$$\int_0^{h/2} |f(x)\, f'(x)|\, dx \leq \frac{h}{4} \int_0^{h/2} f'(x)^2\, dx,$$

$$\int_0^{h/2} |f(h - x)\, f'(h - x)|\, dx \leq \frac{h}{4} \int_0^{h/2} f'(h - x)^2\, dx.$$

Putting in the second inequality $h - x = t$, and adding the obtained inequality to the first, we get (1).

Besides OLECH's proof, simpler proofs of inequality (1) were given by P. R. BEESACK [3], N. LEVINSON [4], C. L. MALLOWS [5] and R. N. PEDERSEN [6].

We shall give MALLOWS' proof of inequality (2), which according to the above, proves (1).

Put

$$h(x) = \int_0^x |g'(t)|\, dt \qquad (0 \leq x \leq a).$$

Then, for $0 \leq x \leq a$, $|g(x)| \leq h(x)$, and so

$$2 \int_0^a |g'(x)\, g(x)|\, dx \leq 2 \int_0^a h'(x)\, h(x)\, dx = h(a)^2.$$

However, according to the BUNIAKOWSKI-SCHWARZ inequality we have

$$h(a)^2 = \left(\int_0^a h'(x)\, dx\right)^2 \leq \int_0^a dx \int_0^a h'(x)^2\, dx = a \int_0^a |g'(x)|^2\, dx. \tag{3}$$

Equality holds in (3), and therefore in (2), if and only if $g(x) = cx$ (c is a constant).

P. R. BEESACK has, in the cited paper [3], proved among other things, the following two theorems:

Theorem 3. *Let p be a positive and continuous function on (a, X) $(-\infty \leq a < X < +\infty)$ and let $\int_a^X \frac{1}{p(x)}\, dx < +\infty$. Furthermore, let f be absolutely continuous on $[a, X]$ and let*

$$f(x) = \int_a^x f'(t)\, dt \quad (a \leq x \leq X) \quad \text{and} \quad f(x)^2 = O\left(\int_a^x \frac{dt}{p(t)}\right) \quad (x \to a+).$$

Then the following inequality holds

$$\int_a^X |f(x)\, f'(x)|\, dx \leq \frac{1}{2} \int_a^X \frac{dx}{p(x)} \int_a^X p(x)\, f'(x)^2\, dx,$$

with equality if and only if

$$f(x) = c \int_a^x \frac{dt}{p(t)},$$

where c is a constant.

Theorem 4. *Let p be a positive and continuous function on (a, b) and let $\int_a^b \frac{dx}{p(x)} < +\infty$. Let f be absolutely continuous on every subinterval*

$[a, X]$, $[X, b]$ and let

$$f(x)=\int_a^x f'(t)\,dt\quad (a\le x\le X),\qquad f(x)=-\int_x^b f'(t)\,dt\quad (X\le x\le b),$$

$$f(x)^2=O\left(\int_a^x \frac{dt}{p(t)}\right)\ (x\to a+),\qquad f(x)^2=O\left(\int_x^b \frac{dt}{p(t)}\right)\ (x\to b-).$$

Then

$$\int_a^b |f(x)\,f'(x)|\,dx\le \frac{K}{2}\int_a^b p(x)\,f'(x)^2\,dx, \tag{4}$$

where X and K are determined by

$$\int_a^X \frac{dt}{p(t)}=\int_X^b \frac{dt}{p(t)}=K.$$

Equality holds in (4) if and only if

$$\begin{aligned} f(x)&=A\int_a^x \frac{dt}{p(t)} && (a\le x<X),\\ &=B\int_x^b \frac{dt}{p(t)} && (X<x\le b).\end{aligned}$$

Opial's inequality results from Theorem 4 by putting $a=0$ and $p(x)=1$.

Beesack's proof of these theorems has been simplified by G.-S. Yang [7]. He has, at the same time, obtained a generalization of Theorem 3.

Theorem 5. *Let p be a positive function on $[a, X]$ and $\int_a^X \frac{dx}{p(x)}<+\infty$, and let q be positive, bounded and nonincreasing on $[a, X]$. If f is absolutely continuous on $[a, X]$ and $f(a)=0$, then*

$$\int_a^X q(x)\,|f(x)\,f'(x)|\,dx\le \frac{1}{2}\int_a^X \frac{dx}{p(x)}\int_a^X p(x)\,q(x)\,f'(x)^2\,dx,$$

with equality if and only if

$$q(x)=\text{const}\quad \textit{and}\quad f(x)=c\int_a^x \frac{dt}{p(t)}\qquad (c=\text{const}).$$

A generalization of Theorem 4 can also be obtained from the above theorem.

A generalization in the other direction of Opial's inequality is contained in the following theorem:

Theorem 6. *If f is absolutely continuous on $[a, b]$ and $f(a)=0$, then for all $p\ge 0$, $q\ge 1$,*

$$\int_a^b |f(x)|^p\,|f'(x)|^q\,dx\le \frac{q}{p+q}(b-a)^p\int_a^b |f'(x)|^{p+q}\,dx. \tag{5}$$

Inequality (5) for $q = 1$ and p a natural number, was first shown by L.-K. HUA [8]. For $q = 1$ and $p \geq 0$, inequality (5) can be obtained as a special case of a more general result of J. CALVERT [9]. In this case, a very short proof was given by J. S. W. WONG [10]. Inequality (5) has been proved by G.-S. YANG [7] for $p, q \geq 1$, but the proof also holds for $p \geq 0$, $q \geq 1$, as noticed by P. R. BEESACK.

If $f(a) = f(b) = 0$, then Theorem 6 yields

Theorem 7. *If f is absolutely continuous on $[a, b]$ and $f(a) = f(b) = 0$, then for all $p \geq 0$ and $q \geq 1$, we have*

$$\int_a^b |f(x)|^p \, |f'(x)|^q \, dx \leq \frac{q}{p+q} \left(\frac{b-a}{2}\right)^p \int_a^b |f'(x)|^{p+q} \, dx. \tag{6}$$

The above results were generalized by P. R. BEESACK and K. M. DAS [11]. We shall only quote the following result:

Theorem 8. *Let p, q be real numbers such that $pq > 0$ and either $p + q > 1$, or $p + q < 0$ and let r, s be nonnegative measurable functions on (a, X) such that $\int_a^X r(x)^{-1/(p+q-1)} \, dx < +\infty$ and the constant K_1 defined by*

$$K_1(X, p, q) = \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \left(\int_a^X s(t)^{\frac{p+q}{p}} \, r(t)^{-\frac{q}{p}} \left(\int_a^t r(u)^{\frac{-1}{p+q-1}} \, du\right)^{p+q-1} dt\right)^{\frac{p}{p+q}}$$

is finite, where $-\infty \leq a < X \leq +\infty$. If f is absolutely continuous on $[a, X]$, $f(a) = 0$ and f' does not change the sign on (a, X), then

$$\int_a^X s(x) \, |f(x)|^p \, |f'(x)|^q \, dx \leq K_1(X, p, q) \int_a^X r(x) \, |f'(x)|^{p+q} \, dx.$$

Equality holds if and only if either $q > 0$ and $f \equiv 0$, or

$$s(x) = k_1 r(x)^{\frac{q-1}{p+q-1}} \left(\int_a^x r(t)^{\frac{-1}{p+q-1}} \, dt\right)^{\frac{p(1-q)}{q}},$$

and

$$f(x) = k_2 \int_a^x r(t)^{\frac{-1}{p+q-1}} \, dt$$

for some constants k_1 (≥ 0), k_2 real.

This paper of P. R. BEESACK and K. M. DAS contains other results on inequalities of the form

$$\int_a^b s(x) \, |f(x)|^p \, |f'(x)|^q \, dx \leq K(p, q) \int_a^b r(x) \, |f'(x)|^{p+q} \, dx,$$

which we shall, owing to the lack of space, not expose here.

J. CALVERT has, in the quoted paper [9], proved the following two theorems which generalize OPIAL's inequality:

Theorem 9. *Let f be absolutely continuous on (a, b) with $f(a) = 0$, where $-\infty \leq a < b < +\infty$. Let g be a continuous, complex function defined for all t in the range of f and for all real t of the form $t = \int_a^X |f'(u)|\,du$. Suppose that $|g(t)| \leq g(|t|)$ for all t and $g(t_1) \leq g(t_2)$ for $0 \leq t_1 \leq t_2$. Let r be positive, continuous and in $L^{1-q}[a, b]$, where $p^{-1} + q^{-1} = 1$ $(p > 1)$. If $F(x) = \int_0^x g(t)\,dt$ for $x > 0$, then*

$$\int_a^b |g(f)\, f'(x)|\,dx \leq F\left(\left(\int_a^b r(x)^{1-q}\,dx\right)^{1/q}\left(\int_a^b r(x)\,|f'(x)|^p\,dx\right)^{1/p}\right),$$

with equality if $f(x) = c\int_a^x r(t)^{1-q}\,dt$.

The same result (but with equality for $f(x) = c\int_x^b r(t)^{1-q}\,dt$) holds if $f(b) = 0$, where $-\infty < a < b \leq +\infty$.

Theorem 10. *Let f, g, r be as in Theorem 9. If $p < 1$, $1/p + 1/q = 1$, $-\infty \leq a < b < +\infty$ and $f(a) = 0$ and $G(x) = \int_0^x \frac{dt}{g(t)}$, then*

$$\int_a^b \left|\frac{f'(x)}{g(f)}\right| dx \geq G\left(\left(\int_a^b r(x)^{1-q}\,dx\right)^{1/q}\left(\int_a^b r(x)\,|f'(x)|^p\,dx\right)^{1/p}\right),$$

with equality if $f(x) = c\int_a^x r(x)^{1-q}\,dx$.

A corresponding result holds if $f(b) = 0$, $-\infty < a < b \leq +\infty$.

As noticed by P. R. Beesack, the conditions for equality in the above theorems are only sufficient, but not necessary.

The following theorem of D. W. Boyd and J. S. W. Wong [12] bears some relation to Theorem 9.

Theorem 11. *Let ω and σ be nonnegative continuously differentiable real functions on $[0, a]$ $(0 < a < +\infty)$, such that the boundary problem*

$$\frac{d}{dt}\left(\sigma(t)\, u'(t)^p\right) = \lambda\omega'(t)\, u(t)^p, \tag{7}$$

$$u(0) = 0, \quad \sigma(a)\, u'(a)^p = \lambda\omega(a)\, u(a)^p, \quad u' > 0 \text{ on } [0, a]$$

has a solution.

If f is an absolutely continuous function on $[0, a]$ and $f(0) = 0$, then the following inequality holds for all $p > 0$

$$\int_0^a |f'(t)\, f(t)^p|\,\omega(t)\,dt \leq \frac{1}{\lambda_0(p+1)}\int_0^a |f'(t)|^{p+1}\,\sigma(t)\,dt,$$

where λ_0 is the largest positive eigenvalue of the boundary problem (7).

Remark. In the formulation of Theorem 11 we have taken into account a comment of J. V. RYFF in Math. Reviews **35**, 563 (1968).

OPIAL's inequality is obtained for $\sigma = \omega = 1$, $p = 1$.

Results analogous to CALVERT's were obtained by E. K. GODUNOVA and V. I. LEVIN [13]:

Theorem 12. *Let f be absolutely continuous on $[a, b]$ with $f(a) = f(b) = 0$. Let, further, $q(x) > 0$ for $x \in [a, b]$ and $\int_a^b q(x)\,dx = 1$ and let φ and ψ be convex increasing functions for $x > 0$ and $\varphi(0) = 0$. Then we have*

$$\int_a^b \varphi'(|f(x)|)\,|f'(x)|\,dx \leq 2\varphi\left(\psi^{-1}\left(\int_a^b q(x)\,\psi\left(\frac{|f'(x)|}{2q(x)}\right)dx\right)\right). \tag{8}$$

Theorem 13. *Let f be an absolutely continuous function on $[a, b]$ with $f(a) = 0$. If φ is a convex increasing function for $x \geq 0$ and $\varphi(0) = 0$, then*

$$\int_a^b \varphi'(|f(x)|)\,|f'(x)|\,dx \leq \varphi\left(\int_a^b |f'(x)|\,dx\right).$$

Putting in (8) $\varphi(t) = \frac{t^2}{2}$, $\psi(t) = t^p$, $p > 1$, and

$$q(x) = s(x)^{\frac{1}{1-p}}\left(\int_a^b s(t)^{\frac{1}{1-p}}\,dt\right)^{-1},$$

where $s(t) \geq 0$ for $t \in [a, b]$, we get

$$\int_a^b |f(x)\,f'(x)|\,dx \leq C\left(\int_a^b s(x)\,|f'(x)|^p\,dx\right)^{2/p}, \tag{9}$$

where

$$C = \frac{1}{4}\left(\int_a^b s(t)^{\frac{1}{1-p}}\,dt\right)^{\frac{2(p-1)}{p}}.$$

Inequality (9) has been proved by P. MARONI [14] but only for $1 < p \leq 2$, but instead of C, with a weaker constant $C \cdot 2^{2/p-1}$.

The following generalization of OPIAL's inequality is due to R. REDHEFFER [15]:

Theorem 14. *If u, v, w, f are absolutely continuous functions on $[a, b]$ and*

$$\frac{w'(x)}{v(x)}\,f(x)^2 + \frac{w(x)}{v'(x)}\,f'(x)^2 \leq 0, \quad w(x)\,v'(x) \geq 0, \quad v'(x) \neq 0,\ v(x) > 0,$$

then

$$\int_a^b \frac{w(x)}{v'(x)}\left[f(x)^2\,u'(x)^2 + \frac{1}{2}\left(f(x)^2\right)'\left(u(x)^2\right)'\right]dx \geq B - A,$$

where

$$B = \lim_{x \to b} u(x)^2\,f(x)^2\,\frac{w(x)}{v(x)} \quad \textit{and} \quad A = \lim_{x \to a+} u(x)^2\,f(x)^2\,\frac{w(x)}{v(x)}.$$

Finally, let us mention the following generalization of OPIAL's inequality:

Theorem 15. *Let $f \in C^{n-1}[a, b]$ and let $f^{(n-1)}$ be an absolutely continuous function such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ $(n \geq 1)$. Then there exists a constant c_n such that*

$$\int_a^b |f(x) f^{(n)}(x)|\, dx \leq c_n (b-a)^n \int_a^b |f^{(n)}(x)|^2\, dx. \tag{10}$$

D. WILLETT [16] has proved that $c_n \leq 1/2$. He used inequality (10) to prove the existence and uniqueness of the solution of a linear differential equation of order n.

K. M. DAS [17] improved the estimate given by D. WILLETT. In fact, he proved that $c_n \leq \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2}$ and that this inequality is strict except in the case $n = 1$.

Finally, D. W. BOYD [18] determined the best possible constant in (10), by proving:

For n odd $(= 2m+1)$, the best possible constant is $\lambda_0/2^n(n-1)!$, where λ_0 is the largest positive eigenvalue of the following $(m+1) \times (m+1)$ matrix

$$A = (a_{ij}), \text{ with } a_{ij} = \binom{2m}{2i}(2m-2j+2i+1)^{-1} \qquad (i, j = 0, 1, \ldots, m).$$

For n even $(= 2m)$, the best possible constant c_n is $(\alpha_0^n 2^n (n-1)!)^{-1}$, where α_0 is the smallest positive solution of the equation $\det B(\alpha) = 0$ and where $B(\alpha) = (b_{ij}(\alpha))$ is the $m \times m$ matrix defined by

$$b_{ij}(\alpha) = \omega^{2ij}\alpha^{2i} \sum_{k=0}^{2m-2i-1} (-1)^k \frac{g_j^{(k)}(1)}{k} \qquad (i, j = 0, \ldots, m-1),$$

with

$$g_j(x) = \cosh(\omega^j \alpha x) \quad \text{and} \quad \omega = \exp\frac{2\pi i}{n}.$$

In the same article D. W. BOYD has proved the following result:

Theorem 16. *Let c_n be the best constant in* (10). *Then*

$$c_n = \frac{b_n}{2n!}, \quad \textit{where} \quad \frac{1}{2} < b_n \leq \left(\frac{n}{4n-2} + \binom{2n}{n}^{-1}\right)^{1/2}$$

so $b_n \to 1/2$ as $n \to +\infty$.

D. W. BOYD in paper [19] connected the inequalities of WIRTINGER and of Z. OPIAL, by proving an inequality which contains, as special cases, both quoted inequalities. Concerning this, see (12) in 2.23.1.

J. M. HOLT [20] generalized the inequality of P. R. BEESACK which appears in Theorem 3. His result is too complicated to be exposed here.

Discrete analogues of OPIAL's inequality and its generalizations were given by J. S. W. WONG, CHENG-MING LEE and P. R. BEESACK.

J. S. W. WONG [10] has proved the following analogue of the inequality of L. K. HUA, i.e., inequality (5) for $q = 1$.

Theorem 17. *Let u_i be a nondecreasing sequence of nonnegative real numbers. Then, for $p \geq 1$, we have*

$$\sum_{i=1}^{n} (u_i - u_{i-1}) u_i^p \leq \frac{(n+1)^p}{p+1} \sum_{i=1}^{n} (u_i - u_{i-1})^{p+1} \quad (u_0 = 0). \tag{11}$$

CHENG-MING LEE [21] has given a discrete analogue of inequality (5). This inequality includes the result of J. S. W. WONG.

Theorem 18. *Let u_i be a nondecreasing sequence of nonnegative real numbers. If $p, q > 0$, $p + q \geq 1$ or $p, q < 0$, we have*

$$\sum_{i=1}^{n} (u_i - u_{i-1})^q u_i^p \leq K_n \sum_{i=1}^{n} (u_i - u_{i-1})^{p+q} \quad (u_0 = 0),$$

where

$$K_0 = \frac{q}{p+q} \quad and \quad K_n = \max\left(K_{n-1} + \frac{pn^{p-1}}{p+q}, \frac{q(n+1)^p}{p+q}\right) \quad (n = 1, 2, \ldots).$$

If $p > 0$, $q < 0$, $p + q \leq 1$, $p + q \neq 0$ or $p < 0$, $q > 0$, $p + q \geq 1$, then

$$\sum_{i=1}^{n} (u_i - u_{i-1})^q u_i^p \geq C_n \sum_{i=1}^{n} (u_i - u_{i-1})^{p+q} \quad (with\ u_0 = 0),$$

where

$$C_0 = \frac{q}{p+q} \quad and \quad C_n = \min\left(C_{n-1} + \frac{pn^{p-1}}{p+q}, \frac{q(n+1)^p}{p+q}\right) \quad (n = 1, 2, \ldots).$$

If $p, q \geq 1$, CHENG-MING LEE has proved that

$$K_n = \frac{q(n+1)^p}{p+q};$$

if $p \leq 0$, $q < 0$, then

$$K_1 = 1, \quad \text{and} \quad K_n = 1 + \frac{p}{p+q} \sum_{i=2}^{n} i^{p-1} \quad (n = 2, 3, \ldots);$$

if $p \geq 0$, $p + q < 0$, then

$$C_1 = 1 \quad \text{and} \quad C_n = 1 + \frac{p}{p+q} \sum_{i=2}^{n} i^{p-1} \quad (n = 2, 3, \ldots).$$

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2.24 Inequalities Connected with Majorization of Vectors

Definition 1. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. A vector $y = (y_1, \ldots, y_n)$ is said to be majorized by a vector $x = (x_1, \ldots, x_n)$, in symbols $x \succ y$ or $y \prec x$, if, after possible reordering of its components so that*

$$x_1 \geq \cdots \geq x_n, \quad \textit{and} \quad y_1 \geq \cdots \geq y_n,$$

we have

(1) $$\sum_{r=1}^{k} x_r \geq \sum_{r=1}^{k} y_r \quad \textit{for} \quad k = 1, \ldots, n-1$$

and

(2) $$\sum_{r=1}^{n} x_r = \sum_{r=1}^{n} y_r .$$

If condition (2) *is replaced by*

(3) $$\sum_{r=1}^{n} x_r \geq \sum_{r=1}^{n} y_r ,$$

we write $x \gg y$ *or* $y \ll x$.

Definition 2. *Let* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *be two real vectors. If, for* $k = 1, \ldots, n$, $x_k > y_k$ *we write* $x > y$.

Definition 3. *Let* F *be a real function of an* n*-dimensional vector variable. We say that* F *is a symmetric gauge function if it satisfies:*

$$F(x) > 0 \quad \textit{for} \quad x \neq 0;$$

$$F(tx) = |t| \, F(x);$$

$$F(x+y) \leq F(x) + F(y);$$

$$F(x_\pi) = F(x);$$

$$F(xJ) = F(x),$$

where x *and* y *are arbitrary vectors,* t *is a real number,* x_π *denotes the vector obtained by permuting the components of* x, *and where* J *is an arbitrary diagonal matrix whose elements are* $+1$ *or* -1.

Definition 4. *Let the function* $x \mapsto f(x)$ *be nonnegative and integrable on* $(0, 1)$ *so that it is measurable and finite almost everywhere and let* $\mu(s)$ *be the measure of the set on which* $f(x) \geq s$. *The function* $x \mapsto f^*(x)$ *which is inverse to* μ *is called the decreasing rearrangement of* f.

Definition 5. *If* $x, y \in L^1(0, 1)$, *we say that* y *majorizes* x, *in writing* $x \prec y$, *if*

$$\int_0^s x^*(t)\, dt \leq \int_0^s y^*(t)\, dt \qquad \textit{for} \ 0 < s < 1,$$

and

$$\int_0^1 x(t)\, dt = \int_0^1 y(t)\, dt .$$

G. H. Hardy, J. E. Littlewood and G. Pólya [1] proved in 1929 the following:

Theorem 1. *A necessary and sufficient condition in order that*

$$\sum_{r=1}^{n} f(x_r) \leq \sum_{r=1}^{n} f(y_r) \tag{4}$$

holds for every convex function f on I, with $x_r, y_r \in I$, is that $x \prec y$.

This inequality is a generalization of Jensen's inequality for convex functions. Indeed, putting in (4) $x_1 = \cdots = x_n = \frac{1}{n} \sum_{r=1}^{n} y_r$, we get Jensen's inequality.

Proof. Functions f and g defined by $f(x) = x$ and $g(x) = -x$ are convex on an arbitrary segment. Thus

$$\sum_{r=1}^{n} x_r \leq \sum_{r=1}^{n} y_r \quad \text{and} \quad \sum_{r=1}^{n} (-x_r) \leq \sum_{r=1}^{n} (-y_r),$$

whence

$$\sum_{r=1}^{n} x_r = \sum_{r=1}^{n} y_r .$$

The function f defined by

$$f(x) = 0 \text{ for } x \leq y_r \text{ and } f(x) = x - y_r \text{ for } x > y_r$$

is positive and convex on an arbitrary segment, and $f(x) \geq x - y_r$.

Therefore

$$x_1 + \cdots + x_r - r y_r \leq \sum_{i=1}^{r} f(x_i) \leq \sum_{i=1}^{r} f(y_i) = y_1 + \cdots + y_r - r y_r ,$$

from which follows $x \prec y$.

So the condition $x \prec y$ is necessary. Let us now prove that it is also sufficient.

For any convex function f, for $x_1 > y_1$, $x_2 > y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, we have

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq \frac{f(y_1) - f(y_2)}{y_1 - y_2} .$$

Denote

$$D_k = \frac{f(x_k) - f(y_k)}{x_k - y_k} \qquad (x_k \neq y_k) .$$

Then $D_k \geq D_{k+1}$, and in virtue of $x \prec y$ we have

$$\sum_{k=1}^{n-1} (X_k - Y_k)(D_k - D_{k+1}) + (X_n - Y_n) D_n \leq 0, \tag{5}$$

where

$$X_k = \sum_{r=1}^{k} x_r \quad \text{and} \quad Y_k = \sum_{r=1}^{k} y_r.$$

From (5) follows

$$\sum_{k=1}^{n} X_k (D_k - D_{k+1}) + X_n D_n \le \sum_{k=1}^{n} Y_k (D_k - D_{k+1}) + Y_n D_n,$$

i.e.,

$$\sum_{k=1}^{n} (X_k - X_{k-1})\, D_k \le \sum_{k=1}^{n} (Y_k - Y_{k-1})\, D_k.$$

Since $X_k - X_{k-1} = x_k$ and $Y_k - Y_{k-1} = y_k$, from the last inequality we get (4).

Therefore, the condition $x \prec y$ is also sufficient, which completes the proof of Theorem 1.

The proof of necessity of $x \prec y$ is taken from the mentioned paper [1]. The proof of sufficiency is due to L. Fuchs [2].

In 1932 J. Karamata [3] rediscovered Theorem 1.

We shall now quote some results which extend Theorem 1.

Theorem 2. *A necessary and sufficient condition for inequality* (4) *to hold for every convex and increasing function f is $x \ll y$.*

This theorem was proved in 1949 by M. Tomić [4], and in 1950 by G. Pólya [5]. H. Weyl [6] in the same year as M. Tomić has obtained Theorem 2 for $x > 0$ and $y > 0$, while M. Tomić did not require these restrictions.

M. Tomić proved that Theorem 2 contains in a certain sense Theorem 1. He gave a geometrical proof for this, basing it on Gauss' theorem about the centroid [7]. G. Pólya proved Theorem 2 starting with Theorem 1.

L. Fuchs [2] gave the following generalization of Theorem 1.

Theorem 3. *Inequality*

$$\sum_{r=1}^{n} p_r f(x_r) \le \sum_{r=1}^{n} p_r f(y_r)$$

holds for every convex function f and for arbitrary real numbers $p_1, \ldots, p_n$ if and only if

$$x_1 \ge \cdots \ge x_n,\ \ y_1 \ge \cdots \ge y_n, \quad \sum_{r=1}^{k} p_r x_r \le \sum_{r=1}^{k} p_r y_r \qquad (k = 1, \ldots, n-1)$$

and

$$\sum_{r=1}^{n} p_r x_r = \sum_{r=1}^{n} p_r y_r.$$

T. Popoviciu in [8], [9] and [10] generalized Theorem 1 for the functions which are convex of order n.

Theorem 4. *The inequality*

$$\sum_{r=1}^{m} p_r f(x_r) \geq 0$$

holds for all $x_1 \leq \cdots \leq x_m$, $p_i \neq 0$ $(i = 1, \ldots, m)$ *and for every function* f *which is convex of order* n, *if and only if*

$$\sum_{r=1}^{m} p_r x_r^k = 0 \quad \text{for} \quad k = 0, 1, \ldots, n,$$

and if for $t \in (x_k, x_{k+1})$ *and* $k = 1, \ldots, m - n - 1$,

$$-\sum_{r=1}^{k} p_r (x_r - t)^n = \sum_{r=k+1}^{m} p_r (x_r - t)^n \geq 0.$$

K. Fan [11] has proved the following theorem.

Theorem 5. *If* $x \geq 0$ *and* $y \geq 0$, *then* $x \prec y$ *is the necessary and sufficient condition so that for every symmetric gauge function* F *we have*

$$F(x) \leq F(y).$$

L. Mirsky [12] gave an elegant proof of Theorem 5.

In the further text we shall use the following notations:

S the set of symmetric functions,

I the set of increasing functions,

L the set of functions F such that $F(x\Delta) \leq F(x)$, where x is an arbitrary vector and Δ an arbitrary diagonal matrix whose elements are 0 or 1,

C the set of convex functions,

G the set of symmetric gauge functions.

The following three theorems are due to L. Mirsky [12].

Theorem 6. *The inequality*

$$F(x) \leq F(y) \tag{6}$$

holds for any function $F \in C \cap S$ *if and only if* $x \prec y$.

L. Mirsky deduced from this theorem Theorem 1 as a special case.

Theorem 7. *Inequality* (6) *holds for any function* $F \in C \cap S \cap I$ *if and only if* $x \ll y$.

From this theorem the cited results of M. Tomić and G. Pólya can be obtained, as well as Theorem 5 of K. Fan.

Theorem 8. *The inequality* (6) *holds for any function* $F \in C \cap S \cap L$ *if and only if*

$$x \ll y^+, \qquad -x \ll (-y)^+,$$

where $x^+ = (x_1^+, \ldots, x_n^+)$ *and* $x_k^+ = \max(x_k, 0)$.

The following four theorems are also connected with majorization.

Theorem 9. *If* $a_1 > 0, \ldots, a_n > 0$, $b_1 \geq \cdots \geq b_n > 0$, *and* $b_1/a_1 \leq \cdots \leq b_n/a_n$, *then* f, *defined by*

$$f(r) = \left(\frac{a_1^r + \cdots + a_n^r}{b_1^r + \cdots + b_n^r}\right)^{1/r},$$

is an increasing function.

The above theorem was proved by A. W. Marshall, I. Olkin and F. Proschan [13].

Theorem 10. *Let* $X_1, \ldots, X_n$ *be random variables such that their joint distribution is invariant under permutations of its arguments. If* $a \succ b$, *and if* F *is a convex and a symmetric function, then*

$$EF(a_1X_1, \ldots, a_nX_n) \geq EF(b_1X_1, \ldots, b_1X_1),$$

where E *denotes the mathematical expectation.*

This theorem of A. W. Marshall and F. Proschan [14] contains Theorem 1 as well as the following theorem of R. F. Muirhead [15].

Theorem 11. *If* $y_k > 0$ *for* $k = 1, \ldots, n$ *and if* $a \succ b$, *then*

$$\Sigma!\, y_1^{a_1} \cdots y_n^{a_n} \geq \Sigma!\, y_1^{b_1} \cdots y_n^{b_n}, \tag{7}$$

where $\Sigma!$ *denotes summation over the* $n!$ *permutations of* $y_1, \ldots, y_n$.

More generally,

$$\Sigma!\, F(a_1x_1, \ldots, a_nx_n) \geq \Sigma!\, F(b_1x_1, \ldots, b_nx_n),$$

where F *is a convex symmetric function and where* $\Sigma!$ *denotes summation over the* $n!$ *permutations of* $x_1, \ldots, x_n$.

Muirhead's inequality is considered in detail in the book of G. H. Hardy, J. E. Littlewood and G. Pólya [16], pp. 44—49. Also see the paper [17] of R. Rado, and the paper [18] of G. A. Bekišev.

Definition 6. *A real function* F *of* n *real variables is called* Schur's *function, if for all* $i \neq j$,

$$(x_i - x_j)\left(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}\right) \geq 0.$$

Remark. In the above definition it was supposed that F is a differentiable function. However, a definition of SCHUR's functions which are not differentiable is also given in the literature.

A. OSTROWSKI [19] has proved the following result:

Theorem 12. *Inequality*

$$F(x) \geq F(y)$$

holds for any SCHUR's *function F if and only if $x \succ y$.*

We shall now quote some integral inequalities which are connected with the majorization of functions.

G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA proved in [1] an integral analogue of the inequality which appears in Theorem 1.

The following theorem is due to J. V. RYFF [20].

Theorem 13. *Let x and y be bounded and measurable functions on* $[0, 1]$. *The inequality*

$$\int_0^1 \log\left(\int_0^1 u(t)^{x(s)}\,dt\right) ds \leq \int_0^1 \log\left(\int_0^1 u(t)^{y(s)}\,dt\right) ds \tag{8}$$

holds for any positive function u such that $u^p \in L^1$ for any finite p, if and only if $x \prec y$.

Inequality (8) represents an integral analogue of MUIRHEAD's inequality (7).

Finally, we shall quote the following result due to G. F. D. DUFF [24].

Let the sequence $a = (a_1, \ldots, a_n)$ be rearranged as $a^* = (a_1^*, \ldots, a_n^*)$, with $a_1^* \geq \cdots \geq a_n^*$. Then

$$\sum_{k=1}^{n-1} |\Delta a_k^*|^p \leq \sum_{k=1}^{n-1} |\Delta a_k|^p, \tag{9}$$

where $p \geq 1$ and $\Delta a_k = a_{k+1} - a_k$, $\Delta a_k^* = a_{k+1}^* - a_k^*$.

Equality in (9) holds if and only if $a_k = a_k^*$ for $k = 1, \ldots, n$.

The continuous analogue of (9) is

$$\int_a^b |f^{*\prime}(x)|^p\,dx \leq \int_a^b |f'(x)|^p\,dx,$$

where $p \geq 1$ and where f^* is the decreasing rearrangement of f.

DUFF also proved other interesting results, but J. V. RYFF in Math. Reviews **37**, 69 (1969), has given some critical comments on the rigour of the proof.

The question of priority of Theorem 1. From the above exposition follows that Theorem 1 is due to G. H. Hardy, J. E. Littlewood and G. Pólya [1]. However, in the literature one can find that Theorem 1 is also ascribed to J. Karamata.

In the book [21] of E. F. Beckenbach and R. Bellman the titles (pp. 30 and 31) concerning Theorem 1 are "An Inequality of Karamata" and "Proof of the Karamata Result".

In the expository article [22], p. 20, of N. G. de Bruijn Theorem 1 is again ascribed to J. Karamata [3], H. Weyl [6] and G. Pólya [5]. However, G. Pólya has quoted in [5] that the result in question of Weyl [6] is similar to [1] which is due to G. H. Hardy, J. E. Littlewood and G. Pólya, published in 1929 and rediscovered in 1932 by J. Karamata [3].

A. Ostrowski in [19], p. 261, L. Fuchs in [2], p. 53, L. Mirsky in [23], p. 159, acknowledged the priority of Theorem 1 to G. H. Hardy, J. E. Littlewood and G. Pólya.

In Zentralblatt für Mathematik **5**, 201 (1933), W. Fenchel in his review of Karamata's paper [3] has written: „In einem Zusatz wird darauf hingewiesen, daß sich das obige Resultat schon in einer Note von Hardy, Littlewood and Pólya [Messenger Math. **58**, 149–152 (1929)] findet". However, we cannot find in [3] such an addition.

W. W. Rogosinski in Jahrbuch über die Fortschritte der Mathematik **58**, 211 (1932), for the same paper, at the end of his review draws attention to [1] without further comments.

It is interesting to notice that in [4] M. Tomić did not attribute sufficient importance to his result cited above as Theorem 2, which was discovered by H. Weyl [6] at the same time, but with less generality. Tomić's paper was written in Serbian with summaries in French and Russian. However, in these summaries Theorem 2 is not cited.

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2.25 Inequalities for Vector Norms

2.25.1 Triangle Inequality

Definition 1. *Let R be the set of all real numbers and R^m the set of all ordered m-tuples $a = (a_1, \ldots, a_m)$ of real numbers.*

The function

$$(1) \qquad (a, b) \mapsto a \cdot b = a_1 b_1 + \cdots + a_m b_m$$

from $R^m \times R^m$ into R is called the scalar (or inner) product on R^m. The function

$$(2) \qquad a \mapsto |a| = \sqrt{a \cdot a} = \sqrt{a_1^2 + \cdots + a_m^2}$$

is called the Euclidean norm on R^m.

Theorem 1. *For any $a, b \in R^m$,*

$$(3) \qquad |a + b| \leq |a| + |b|,$$

equality holding if and only if $a = 0$ or $b = ta$ with $t \geq 0$.

Proof. By the CAUCHY inequality we have

$$(4) \qquad (\Sigma a_k b_k)^2 \leq (\Sigma a_k^2) \cdot (\Sigma b_k^2),$$

i.e.,

$$(a \cdot b)^2 \leq |a|^2 \, |b|^2.$$

Now,

$$(5) \qquad (a + b) \cdot (a + b) = a \cdot a + b \cdot b + 2a \cdot b$$

$$\leq a \cdot a + b \cdot b + 2 \, |a| \, |b| = (|a| + |b|)^2,$$

from which the triangle inequality (3) follows.

Equality holds in (3) if and only if it holds in (5), i.e., if $a \cdot b = |a| \, |b|$, which together with the corresponding result for the CAUCHY inequality implies that a and b are proportional.

Hence $a \neq 0$ implies $b = ta$ $(t \in R)$. Replacing this in

$$|a| + |b| = |a + b|,$$

we get

$$1 + |t| = |1 + t|,$$

which implies $t \geq 0$.

R^m becomes a vector space by defining

$$(a_1, \ldots, a_m) + (b_1, \ldots, b_m) = (a_1 + b_1, \ldots, a_m + b_m)$$

as addition, and

$$t(a_1, \ldots, a_m) = (ta_1, \ldots, ta_m)$$

as multiplication by a real number t.

If we put

$$d(a, b) = |a - b| \quad \text{for} \quad a, b \in R^m,$$

then the function d has the following properties:

$$d(a, b) \geq 0,$$

$$d(a, b) = 0 \Leftrightarrow a = b,$$

$$d(a, b) \leq d(a, c) + d(c, b) \quad \text{for all} \quad a, b, c \in R^m.$$

The last inequality is the usual triangle inequality from Analytic Geometry.

The set R^m together with the function d is called the m-dimensional Euclidean vector space and will be denoted by E^m in the sequel.

2.25.2 An Identity of Hlawka and the Associated Inequality

Now we shall prove that

$$(1) \quad |a| + |b| + |c| - |b + c| - |c + a| - |a + b| + |a + b + c| \geq 0,$$

where a, b, c are arbitrary vectors in E^m.

This inequality follows immediately from the triangle inequality and the following identity

$$\begin{aligned}(2)\qquad &(|a| + |b| + |c| - |b + c| - |c + a| - |a + b| + |a + b + c|)\\ &\quad\times (|a| + |b| + |c| + |a + b + c|)\\ &= (|b| + |c| - |b + c|)\,(|a| - |b + c| + |a + b + c|)\\ &\quad+ (|c| + |a| - |c + a|)\,(|b| - |c + a| + |a + b + c|)\\ &\quad+ (|a| + |b| - |a + b|)\,(|c| - |a + b| + |a + b + c|),\end{aligned}$$

which is due to E. HLAWKA [1].

The proof of this identity can be obtained by direct verification.

We note that the triangle inequality and HLAWKA's identity hold also in unitary spaces.

It was conjectured [2] that the inequality (1) holds in every real normed space. This conjecture was shown to be true, see for instance [3]. The analogous assertion for complex spaces is not true.

2.25.3 An Inequality of Hornich

Theorem 1. *Let $a \in E^m$ and $a_k \in E^m$ $(k = 1, \ldots, n)$. If these vectors satisfy*

$$(1)\qquad \sum_{k=1}^{n} a_k = -ta \qquad (t \geq 1),$$

then

$$(2)\qquad \sum_{k=1}^{n} (|a_k + a| - |a_k|) \leq (n - 2)\, |a|.$$

If $t < 1$ in (1), then (2) need not necessarily hold.

Proof. From the inequality of HLAWKA we get

$$\begin{aligned}(3)\qquad &\max(|a_i + a| - |a_i| + |a_j + a| - |a_j|)\\ &\leq |a_i + a_j + a| - |a_i + a_j| + |a|,\end{aligned}$$

where the maximum is taken over all pairs of vectors a_i, a_j such that $a_i + a_j = \text{const}$. Using (3) we see that the sum on the left-hand side of (2) does not decrease if we replace the vectors $a_1, a_2, \ldots, a_n$ by the vectors $0, a_1 + a_2, a_3, \ldots, a_n$. In the same way we conclude that the new sum will not decrease if we replace the vectors $0, a_1 + a_2, a_3, \ldots, a_n$ by $0, 0, a_1 + a_2 + a_3, a_4, \ldots, a_n$. By repeated application of this procedure we conclude that

$$(4)\qquad \sum_{k=1}^{n} (|a_k + a| - |a_k|) \leq \sum_{k=1}^{n} (|b_k + a| - |b_k|),$$

where $b_1 = \cdots = b_{n-1} = 0$, $b_n = a_1 + \cdots + a_n$. Putting these b_k into (4), we get

$$\sum_{k=1}^{n} (|a_k + a| - |a_k|) \leq (n-1)|a| + |a_1 + \cdots + a_n + a| - |a_1 + \cdots + a_n|.$$

By application of (1) we obtain that

$$\sum_{k=1}^{n} (|a_k + a| - |a_k|) \leq (n - 1 - |t|)\,|a| + |t - 1|\,|a|.$$

Since $|t-1| - |t| = -1$ $(t \geq 1)$ it follows that HORNICH's inequality (2) is true.

H. HORNICH [1] gave his inequality in a different form which is a special case of (2). We shall still call it HORNICH's inequality.

The lengthy original proof of HORNICH was shortened by R. P. LUČIĆ [4] who used induction on n. The above general form of this inequality and the proof are due to D. Ž. DJOKOVIĆ [5].

2.25.4 Generalizations of Hlawka's Inequality

D. D. ADAMOVIĆ [6] has established the following inequality

$$(1) \qquad (n-2)\sum_{k=1}^{n} |a_k| + \left|\sum_{k=1}^{n} a_k\right| \geq \sum_{1 \leq i < j \leq n} |a_i + a_j| \qquad (n \geq 2),$$

where $a_k \in E^m$. This inequality contains HLAWKA's inequality as a special case when $n = 3$.

D. D. ADAMOVIĆ's proof is based on an identity which is a generalization of the identity (2) in 2.25.2.

In [7] P. M. VASIĆ has, among other things, given a straightforward proof of this inequality by the method of induction.

D. Ž. DJOKOVIĆ [8] has proved the following more general inequality which contains (1) as a special case:

$$(2) \qquad \sum_{1 \leq i_1 < \cdots < i_k \leq n} |a_{i_1} + \cdots + a_{i_k}| \leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n} |a_i| + \left|\sum_{i=1}^{n} a_i\right|\right).$$

The values of n and k are given by $n = 3, 4, \ldots, k = 2, \ldots, n-1$. Independently of D. Ž. DJOKOVIĆ the inequality (2) was proved also by D. M. SMILEY and M. F. SMILEY [2]. Conditions for equality in (2) are given in [2] and [8].

Inequality (2) was used by T. POPOVICIU [9] as a new characterization of continuous convex functions. In fact, he proved the following theorem:

Let $n \geq 3$ be a positive integer and k an integer such that $2 \leq k \leq n-1$. A continuous function f is nonconcave over some interval I if and only if

$$\sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \leq \frac{1}{k}\binom{n-2}{k-2}\left[\frac{n-k}{k-1}\sum_{i=1}^{n} f(x_i) + nf\left(\frac{x_1+\cdots+x_n}{n}\right)\right]$$

holds for any $x_1, \ldots, x_n \in I$.

Let D be an additive Abelian semigroup with neutral element 0 and let $E \subset D$ be a set with the following properties:

1° $0 \in E$;

2° $a_i \in E\ (i=1,\ldots,n) \wedge \sum_{i=1}^{n} a_i \in E \Rightarrow \sum_{\nu=1}^{m} a_{i_\nu} \in E\quad (1 \leq i_1 < \cdots < i_m \leq n)$.

Furthermore, let G be an ordered Abelian group, i.e. this group is endowed, with a relation of total order $<$ with the property:

$$(a, b, c \in G \wedge a < b) \Rightarrow (a + c < b + c).$$

For a function $f\colon E \to G$, denote by $C_{n,k}$ the condition

$$\sum_{1\leq i_1<\cdots<i_k\leq n} f(a_{i_1}+\cdots+a_{i_k}) + \binom{n-1}{k}(k-1)f(0) \leq \binom{n-2}{k-1}\sum_{i=1}^{n} f(a_i) + \binom{n-2}{k-2} f\left(\sum_{i=1}^{n} a_i\right)$$

$$\left(a_i \in E\quad (1 \leq i \leq n),\quad \sum_{i=1}^{n} a_i \in E\right).$$

P. M. Vasić and D. D. Adamović [10] have proved

Theorem 1. *Conditions* $C_{n,k}$ $(2 \leq k < n;\ n \geq 3)$ *are fulfilled if and only if one of them is satisfied, i.e., each one of these conditions is equivalent to all others.*

Inequalities (1) and (2) are contained in this theorem.

Remark. H. Freudenthal [11] has proposed the following problem: If $a_1, \ldots, a_n$ are arbitrary m-dimensional vectors, for what values of n do we have

$$\sum_{i=1}^{n} |a_i| - \sum_{1\leq i<j\leq n} |a_i + a_j| + \sum_{1\leq i<j<k\leq n} |a_i + a_j + a_k| - \cdots + (-1)^{n-1}|a_1 + \cdots + a_n| \geq 0.$$

If $n = 2$ this is the triangle inequality. If $n = 3$ this is the inequality of Hlawka. If $n \geq 4$ it was shown by W. A. J. Luxemburg that this inequality is false. His counterexample is the following:

$$a_i = b\quad (i = 1, \ldots, n-1),\quad a_n = -2b\quad (b \neq 0).$$

The following very general reduction theorem for linear vector inequalities is due to F. W. LEVI [12].

Theorem 2. *Let k_i and p_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, r)$ be real constants. Suppose that for all real numbers $x_1, \ldots, x_r$ we have*

$$\sum_{i=1}^{n} k_i \,|\, p_{i1} x_1 + \cdots + p_{ir} x_r \,| \geq 0. \tag{3}$$

Then

$$\sum_{i=1}^{n} k_i \,|p_{i1} a_1 + \cdots + p_{ir} a_r| \geq 0 \tag{4}$$

holds for arbitrary vectors $a_1, \ldots, a_r \in E^m$ and any positive integer m.

Proof. Let $b \in E^m$ and

$$F(b) = \int |a \cdot b| \, da,$$

where the integral is taken over the unit sphere $|a| = 1$ in E^m. It is evident that $F(b)$ has the properties

$$\begin{aligned} &F(b) = F(c) \quad \text{if} \quad |b| = |c|, \\ &F(tb) = |t| \, F(b) \quad \text{for any real } t, \\ &F(\varepsilon) = p_m > 0 \quad \text{if} \quad |\varepsilon| = 1. \end{aligned} \tag{5}$$

Replace x_i in (3) by the scalar product $a_i \cdot a$, where $a \in E^m$ is also arbitrary. Then we get

$$\sum_{i=1}^{n} k_i \,|(p_{i1} a_1 + \cdots + p_{ir} a_r) \cdot a| \geq 0.$$

By integration over the unit sphere $|a| = 1$, we get

$$\sum_{i=1}^{n} k_i \int |(p_{i1} a_1 + \cdots + p_{ir} a_r) \cdot a| \, da \geq 0.$$

Using (5) we obtain

$$\sum_{i=1}^{n} k_i \,|p_{i1} a_1 + \cdots + p_{ir} a_r| \, p_m \geq 0,$$

which implies (4).

As an application of the reduction theorem F. W. LEVI [12] proved that

$$\Sigma \,|\pm a_1 \pm \cdots \pm a_r| \geq 2 \binom{r-1}{t} \sum_{i=1}^{r} |a_i|, \tag{6}$$

where $t = \left[\frac{r-1}{2}\right]$, $a_1, \ldots, a_r \in E^m$; and the sum on the left-hand side is taken over all 2^r combinations of $+$ and $-$.

It is easy to verify that

$$\binom{r-1}{t}=\binom{r-1}{s}, \quad \text{if} \quad t=\left[\frac{r-1}{2}\right] \quad \text{and} \quad s=\left[\frac{r}{2}\right].$$

In the case $m = 1$, inequality (6) has been rediscovered by M. MARJANOVIĆ [13].

2.25.5 A Steinitz-Gross Result

Let $a_k \in E^m$ $(k = 1, \ldots, n)$ be such that

$$\sum_{k=1}^{n} a_k = 0, \quad |a_k| \leq 1 \quad (1 \leq k \leq n). \tag{1}$$

Let these vectors be arranged so that they form a closed polygon in E^m, namely

$$OA_1A_2 \cdots A_{n-1}O \qquad (\overrightarrow{OA_1} = a_1, \ldots, \overrightarrow{A_{k-1}A_k} = a_k, \ldots, \overrightarrow{A_{n-1}O} = a_n).$$

Let p_m be the least positive number having the following property: Given any system of vectors $a_k \in E^m$ $(k = 1, \ldots, n; n \geq 2)$ satisfying (1); then we can permute $a_2, \ldots, a_n$ so that the corresponding polygon associated with these n vectors arranged in a new order is contained in the sphere $|a| \leq p_m$ with centre at the origin O.

We have evidently $p_1 = 1$. Independently of each other W. GROSS [14], V. BERGSTRÖM [15], and I. DAMSTEEG and I. HALPERIN [16] proved that $p_2 = \sqrt{2}$. The values of p_m for $m > 2$ are not known.

I. DAMSTEEG and I. HALPERIN [16] proved that

$$p_m \geq \frac{1}{2}\sqrt{m+6}\,.$$

F. A. BEHREND [17] has proved that

$$p_m < m, \quad p_3 < \sqrt{5 + 2\sqrt{3}} = 2.90 \ldots$$

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2.26 Mills' Ratio and Some Related Results

The function R defined by

$$R(x) = \frac{\int\limits_x^{+\infty} e^{-t^2/2}\, dt}{e^{-x^2/2}} = e^{x^2/2} \int\limits_x^{+\infty} e^{-t^2/2}\, dt$$

is usually called Mills' ratio.

We shall give here some results which refer to the problem of finding some simple functions which approximate R.

One of the first such results was obtained by R. D. Gordon [1] who proved in 1941 that for all $x > 0$,

$$\frac{x}{x^2+1} \leq R(x) \leq \frac{1}{x}. \tag{1}$$

A year later, Z. W. Birnbaum [2] improved the lower bound of R. D. Gordon, thus obtaining that for all $x > 0$,

$$\frac{1}{2}\left(\sqrt{4+x^2} - x\right) < R(x) \leq \frac{1}{x}. \tag{2}$$

Birnbaum's proof uses properties of convex functions.

Other results were mainly based on improving inequalities (1) and (2) or extending the domain in which they hold.

Thus, for example, M. R. Sampford [3] has proved that, for all real x, $0 < (1/R(x))' < 1$ and $(1/R(x))'' > 0$, and as a consequence that $R(x) < 4/(3x + \sqrt{8 + x^2})$ for $x > -1$.

Sharpening the combined GORDON's and BIRNBAUM's inequalities (2) written in the form

$$(3) \qquad \frac{1}{2\sqrt{2\pi}}\left(\sqrt{4+x^2}-x\right)e^{-x^2/2} \leq \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-t^2/2}\,dt \leq \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,e^{-x^2/2},$$

R. F. TATE [4] has proved that for $x > 0$,

$$(4) \qquad \frac{1}{2}-\sqrt{\frac{1}{4}-\frac{e^{-x^2}}{2}} \leq \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-t^2/2}\,dt \leq \frac{1}{2}+\frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}-\sqrt{\frac{1}{4}+\frac{e^{-x^2}}{2\pi x^2}}.$$

The upper bound in (4) is an improvement on that in (3) for $x > 0$, while the lower bound cannot be compared with BIRNBAUM's since for some values of $x \geq 0$ it is greater and for some smaller than the lower bound in (3).

Y. KOMATU [5] has given an elementary proof of the inequalities

$$(5) \qquad \frac{2}{\sqrt{x^2+4}+x} < R(x) < \frac{2}{\sqrt{x^2+2}+x}.$$

The lower bound in inequality (5) is equal to the lower bound in (2), while the upper bound is an improvement on the result of GORDON.

H. O. POLLAK [6] has shown that the greatest value of β for which

$$R(x) < \frac{2}{\sqrt{x^2+\beta}+x}$$

is $\beta = 8/\pi$. It can easily be shown that the least value of β for which

$$\frac{2}{\sqrt{x^2+\beta}+x} < R(x)$$

is $\beta = 4$, i.e., that BIRNBAUM's lower bound for $R(x)$ is the best possible bound of the form $\frac{2}{\sqrt{x^2+\beta}+x}$. However, $\frac{2}{\sqrt{x^2+4}+x}$ tends to 1 as x tends to 0, whereas $R(x)$ in that case tends to $\sqrt{\pi/2}$. (The upper bound, $\frac{2}{\sqrt{x^2+(8/\pi)}+x}$ also tends to $\sqrt{\pi/2}$ as x tends to 0.)

In order to get lower and upper bounds which have the same limits as $R(x)$, A. V. BOYD [7] tried to find approximations for R in the more general form

$$\frac{\alpha}{\sqrt{x^2+\beta}+\gamma x}.$$

He first noticed that for the approximation to be good for both large and small x, it is necessary that $\alpha = \gamma + 1$, $\beta = 2\alpha^2/\pi$, i.e., that the approximating functions are of the form

$$\frac{\gamma+1}{\sqrt{x^2+(2/\pi)(\gamma+1)^2}+\gamma x}.$$

By expanding $R(x)$ for small and large x, A. V. BOYD succeeded in proving that $\gamma = \pi - 1$ and $\gamma = 2/(\pi - 2)$ give the sharpest lower and upper bounds respectively, thus obtaining for $x > 0$ the inequalities

$$(6) \qquad \frac{\pi}{\sqrt{x^2 + 2\pi} + (\pi - 1)\, x} < R(x) < \frac{\pi}{\sqrt{(\pi - 2)^2 x^2 + 2\pi} + 2x}.$$

Notice that in this case, both bounds in (6) tend to $\sqrt{\pi/2}$ as x tends to 0. The above bounds are better than POLLAK's.

Concerning the function R, see also papers [8]—[11].

Generalizing MILLS' ratio, W. GAUTSCHI [12] has proved that, for $p > 1$ and $0 \leq x < +\infty$,

$$(7) \qquad \frac{1}{2}\left((x^p + 2)^{1/p} - x\right) < e^{x^p} \int_x^{+\infty} e^{-t^p}\, dt \leq C_p \left(\left(x^p + \frac{1}{C_p}\right)^{1/p} - x\right),$$

where $C_p = \left(\Gamma\left(1 + \frac{1}{p}\right)\right)^{\frac{p}{p-1}}$.

A special case of the above inequalities

$$\frac{1}{x + \sqrt{x^2 + 2}} < e^{x^2} \int_x^{+\infty} e^{-t^2}\, dt \leq \frac{1}{x + \sqrt{x^2 + (4/\pi)}}$$

obtained by putting $p = 2$ in (7) is quoted in [13].

A somewhat different line of approach was pursued by R. G. HART, and W. R. SCHUCANY and H. L. GRAY.

R. G. HART defines in [14] the function

$$P(x) = \frac{e^{-x^2/2}}{x}\left(1 - \frac{(1 + bx^2)^{1/2}/(1 + ax^2)}{P_0 x + (P_0^2 x^2 + \exp(-x^2/2)\,(1 + bx^2)^{1/2}/(1 + ax^2))^{1/2}}\right),$$

where $P_0 = \sqrt{\pi/2}$, $a = \frac{1}{2}(1 + (1 - 2\pi^2 + 6\pi)^{1/2})$, $b = 2\pi a^2$, showing that it has a number of properties in common with $F(x) = \int_x^{+\infty} e^{-t^2/2}\, dt$:

1° For all real x, P is real, positive and finite.

2° For all real x, dP/dx is real, negative and finite.

3° For all real x, $P(x) + P(-x) = (2\pi)^{1/2}$.

4° As $x \to 0$, $P(x) \to (\pi/2)^{1/2}$.

5° As $x \to 0$, $dP/dx \to -1$.

6° As $x \to +\infty$, $P \to 0$ and $x \exp(x^2/2)\, P(x) \to 1$.

7° As $x \to +\infty$, $dP/dx \to 0$ and $(d/d(x^{-2}))\,(x \exp(x^2/2)\, P(x)) \to -1$.

Moreover, it was shown that functions P and F have approximately equal values for a number of values of x.

W. R. Schucany and H. L. Gray [15] defined the function E by a simpler expression

$$E(x) = \frac{x \exp(-x^2/2)}{x^2 + 2} \, \frac{x^6 + 6x^4 - 14x^2 - 28}{x^6 + 5x^4 - 20x^2 - x}.$$

Properties analogous to 1°, 2°, 6° and 7° hold for all $x > 2$.

$E(x)$ is a better approximation than $P(x)$, but unfortunately it is not certain whether it holds for $x < 2$ (while $P(x)$ approximates $F(x)$ for all real values of x).

The following results are also related to Mills' ratio:

1° Define I_m by the following expression [16]

$$I_m = \int_0^{+\infty} x^m e^{-nx^2/2} \, dx \quad \text{for} \quad m = 0, 1, \ldots \quad \text{and} \quad n = 1, 2, \ldots$$

Then the following inequality holds

$$I_{n+1} > I_n.$$

2° For all $x > 0$, we have [17], p. 229,

$$e^{-x^2} \int_0^x e^{t^2} \, dt \leq \frac{\pi^2}{8x} (1 - e^{-x^2}).$$

Remark. In Statistics the function R is named after J. P. Mills who has considered it for the first time in [18]. In the literature the name Mills is often wrongly used as Mill. For this information we are obliged to R. F. Tate, Z. W. Birnbaum and C. L. Mallows.

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2.27 Stirling's Formula

We start with the expansion

$$\log \frac{1+x}{1-x} = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right), \tag{1}$$

which holds if $-1 < x < +1$.

For $x = \frac{1}{2n+1}$, with n a positive integer, we get

$$\log(n+1) = \log n + 2\left(\frac{1}{2n+1} + \frac{1}{3}\frac{1}{(2n+1)^3} + \frac{1}{5}\frac{1}{(2n+1)^5} + \cdots\right),$$

i.e.,

$$\left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) = 1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \cdots.$$

Since

$$\begin{aligned} 1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \cdots \\ &< 1 + \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \cdots\right) \\ &= 1 + \frac{1}{12}\frac{1}{n(n+1)} \\ &= 1 + \frac{1}{12}\frac{1}{n} - \frac{1}{12}\frac{1}{n+1}, \end{aligned}$$

we obtain

$$\left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) < 1 + \frac{1}{12}\frac{1}{n} - \frac{1}{12}\frac{1}{n+1}. \tag{2}$$

On the other hand, we have

$$\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) > 1+\frac{1}{3}\,\frac{1}{(2n+1)^2}+\frac{1}{5}\,\frac{1}{(2n+1)^4}$$

$$+\frac{1}{25/3}\,\frac{1}{(2n+1)^6}+\cdots$$

$$=1+\frac{1}{3}\,\frac{1}{(2n+1)^2}\,\frac{1}{1-\frac{1}{5/3}\,\frac{1}{(2n+1)^2}}$$

$$=1+\frac{12}{144n^2+144n+14.4}$$

$$>1+\frac{12}{144n^2+150n+49/16} \qquad (n\geq 2)$$

$$=1+\frac{1}{12n+1/4}-\frac{1}{12(n+1)+1/4}.$$

So we have proved that

$$\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) > 1+\frac{1}{12n+1/4}-\frac{1}{12(n+1)+1/4} \quad (n\geq 2). \tag{3}$$

If $n \geq 2$ from (2) and (3) we get

$$e\exp\left(\frac{1}{12n+1/4}-\frac{1}{12(n+1)+1/4}\right) < \left(1+\frac{1}{n}\right)^{n+1/2} \tag{4}$$

$$< e\exp\left(\frac{1}{12n}-\frac{1}{12(n+1)}\right).$$

Let us now consider the sequence (a_n) of positive numbers

$$a_n=\frac{n!\,e^n}{n^{n+1/2}} \qquad (n\geq 2).$$

Since

$$\frac{a_n}{a_{n+1}}=\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+1/2}, \tag{5}$$

we obtain from (4) that

$$\exp\left(\frac{1}{12n+1/4}-\frac{1}{12(n+1)+1/4}\right) < \frac{a_n}{a_{n+1}} < \exp\left(\frac{1}{12n}-\frac{1}{12(n+1)}\right).$$

It follows that

$$a_{n+1}\exp\left(-\frac{1}{12(n+1)+1/4}\right) < a_n\exp\left(-\frac{1}{12n+1/4}\right)$$

and

$$a_n\exp\left(-\frac{1}{12n}\right) < a_{n+1}\exp\left(-\frac{1}{12(n+1)}\right).$$

Using

$$1<\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right), \text{ i.e., } e<\left(1+\frac{1}{n}\right)^{n+1/2},$$

we get the inequality

$$1 < \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+1/2}.$$

According to (5) this leads to

$$1 < \frac{a_n}{a_{n+1}}, \quad \text{i.e.,} \quad a_{n+1} < a_n.$$

The sequence of positive numbers $a_n \exp\left(-\frac{1}{12n}\right)$ is monotonely increasing and

$$a_n \exp\left(-\frac{1}{12n}\right) < a_n < a_{n-1} < \cdots < \frac{2!\,e^2}{2^{5/2}}.$$

The sequence of positive numbers $a_n \exp\left(-\frac{1}{12n+1/4}\right)$ is monotonely decreasing and

$$a_n > a_n \exp\left(-\frac{1}{12n+1/4}\right) > a_{n+1} \exp\left(-\frac{1}{12n+1/4}\right),$$

i.e.,

$$a_{n+1} \exp\left(-\frac{1}{12n+1/4}\right) < a_n < a_{n-1} < \cdots < \frac{2!\,e^2}{2^{5/2}}.$$

Consequently, we have

$$\lim_{n\to+\infty}\left[a_n \exp\left(-\frac{1}{12n+1/4}\right)\right] = \lim_{n\to+\infty}\left[a_n \exp\left(-\frac{1}{12n}\right)\right] = \lim_{n\to+\infty} a_n = a,$$

where a is a finite positive constant which has to be determined.

We shall use WALLIS' formula

$$\sqrt{\pi} = \lim_{n\to+\infty}\left[\frac{(2n-2)!!}{(2n-1)!!}\,2\sqrt{n}\right] = \lim_{n\to+\infty}\frac{2^{2n} n!\, n!}{(2n)!\sqrt{n}}.$$

By setting $n! = a_n e^{-n} n^{n+1/2}$, we find that

$$\sqrt{\pi} = \lim_{n\to+\infty}\frac{2^{2n} a_n^2 n^{2n+1} e^{-2n}}{(2n)^{2n+1/2} a_{2n} e^{-2n}\sqrt{n}} = \lim_{n\to+\infty}\frac{a_n^2}{a_{2n}\sqrt{2}} = \frac{a}{\sqrt{2}}.$$

Hence, we have proved that

$$a_n \exp\left(-\frac{1}{12n}\right) < a = \sqrt{2\pi} < a_n \exp\left(-\frac{1}{12n+1/4}\right).$$

Further, we obtain

$$\frac{n!\,e^n}{n^{n+1/2}} \exp\left(-\frac{1}{12n}\right) < \sqrt{2\pi} < \frac{n!\,e^n}{n^{n+1/2}} \exp\left(-\frac{1}{12n+1/4}\right),$$

$$\frac{n^{n+1/2}}{n!\,e^n} \exp\frac{1}{12n} > \frac{1}{\sqrt{2\pi}} > \frac{n^{n+1/2}}{n!\,e^n} \exp\frac{1}{12n+1/4},$$

$$(6) \qquad \sqrt{2\pi n}\, n^n e^{-n} \exp\frac{1}{12n+1/4} < n! < \sqrt{2\pi n}\, n^n e^{-n} \exp\frac{1}{12n}.$$

These inequalities hold for $n \geq 2$.

The asymptotic formula

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{7}$$

is due to J. STIRLING (1764).

CESÀRO-BUCHNER's inequalities (6) (cf. [1] — [5]) are more accurate than (7) since they give also lower and upper bounds of $n!$.

The lower bound can be improved, as follows

$$\sqrt{2\pi n}\, n^n e^{-n} \exp \frac{1}{12n + (0.6/n)} < n! \quad (n \geq 3).$$

H. ROBBINS (cf. [6] and [7]) in 1955 proved that

$$\sqrt{2\pi n}\; n^n e^{-n} \exp \frac{1}{12n+1} < n! < \sqrt{2\pi n}\; n^n e^{-n} \exp \frac{1}{12n}.$$

However, these bounds are weaker than those found by P. BUCHNER which were published in 1951.

In a recent paper, P. R. BEESACK [14] used the method of CESÀRO, given above, but carried one additional term in the expansion leading to (2) and (3) to obtain the improved estimates

$$\sqrt{2\pi n}\, n^n e^{-n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3}\right) < n! < \sqrt{2\pi n}\, n^n e^{-n} \exp\left(\frac{1}{12n} - \frac{1}{(360+\gamma_n)\, n^3}\right),$$

where $\gamma_n = 30(7n(n+1)+1)/n^2(n+1)^2$.

W. FELLER [7] mentioned the following asymptotic formulas

$$n! \sim \sqrt{2\pi}\left(n + \frac{1}{2}\right)^{n+1/2} e^{-n-1/2},$$

$$n! \sim \sqrt{2\pi n}\, n^n \exp\left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \cdots\right),$$

and the inequalities

$$\sqrt{2\pi}\left(n + \frac{1}{2}\right)^{n+1/2} e^{-(n+1/2)-(1/24)(n+1/2)} < n! < \sqrt{2\pi}\left(n + \frac{1}{2}\right)^{n+1/2} e^{-(n+1/2)}.$$

Concerning STIRLING's formula there exists an extensive literature. See, in particular, papers [8]—[13].

References

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2. BUCHNER, P.: Bemerkungen zur Stirlingschen Formel. Elem. Math. **6**, 8—11 (1951).
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3. Particular Inequalities

In Part 3 a large number of inequalities, more or less elementary, are included and roughly classified according to the subject matter. A few of these inequalities could also have been incorporated in two or more sections of this Part. All these inequalities can play a certain role in Pure and Applied Mathematics in the proofs of various theorems, or in some other ways.

While in the most cases proofs are omitted because of the lack of space, a very ample bibliography is given as a rule.

Notice that in the references which are cited after almost every inequality, besides the given results there are also other interesting inequalities, but owing to the lack of space they could not be included here.

In the books listed at the end of the Preface and below, other inequalities which do not appear in this Part can also be found.

References

1. PÓLYA, G., and G. SZEGÖ: Aufgaben und Lehrsätze aus der Analysis, vol. 1. Berlin 1925.
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12. OSTROWSKI, A.: Aufgabensammlung zur Infinitesimalrechnung, vol. 1. Basel-Stuttgart 1964.
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Throughout Part 3 we adopt the following abbreviations:
HARDY, LITTLEWOOD, PÓLYA for [1] from the Preface,
OSTROWSKI 1 for [12],

OSTROWSKI 2 for [4],
OSTROWSKI 3 for [7],
MITRINOVIĆ 1 for [10] from the Preface,
MITRINOVIĆ 2 for [12] from the Preface.

3.1 Inequalities Involving Functions of Discrete Variables

3.1.1 Let a, b, r, s and n denote positive integers. Let, furthermore,

$$h(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \qquad g_r(a) = h(ra) - h(r).$$

Then,

$$\frac{a}{a+b} \leq h(a+b) - h(b) < \frac{a}{b},$$

$$0 \leq g_{r+1}(a) - g_r(a) < \frac{1}{r} - \frac{1}{r+1},$$

$$0 \leq g_s(a) - g_r(a) < \frac{1}{r} - \frac{1}{s} \quad \text{if} \quad r < s,$$

$$0 \leq g_r(ab) - g_r(a) - g_r(b) < \frac{1}{r},$$

$$\frac{1}{a+1} \leq g_r(a+1) - g_r(a) < \frac{1}{a}.$$

Reference

LAMBEK, J., and L. MOSER: Rational analogues of the logarithm function. Math. Gaz. **40**, 5–7 (1956).

3.1.2 Let $n > 1$ denote a natural number. Then

$$\log(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \log n.$$

Reference

SCHLÖMILCH, O., and H. LEMONNIER: Problem 455. Nouv. Ann. Math. **18**, 68 and 151 (1859).

3.1.3 If $S_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$, then

$$C < S_p + S_q - S_{pq} \leq 1,$$

where C is EULER's constant.

Reference

SANDHAM, H. F.: Problem E 819. Amer. Math. Monthly **55**, 317 (1948).

3.1.4 If a fixed real number a and a natural number k satisfy the conditions

$$0 \leq a < 1 \quad \text{and} \quad k > \frac{3+a}{1-a},$$

then the following inequality holds for every natural number n:

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{nk-1} > 1 + a.$$

Proof. Denote by $S_k(n)$ the expression on the left side of the last inequality. Using the well-known inequality

$$\frac{1}{a} + \frac{1}{b} > \frac{4}{a+b} \quad (a, b > 0 \text{ and } a \neq b),$$

we get

$$\begin{aligned} 2S_k(n) &= \left(\frac{1}{n} + \frac{1}{nk-1}\right) + \left(\frac{1}{n+1} + \frac{1}{nk-2}\right) + \cdots + \left(\frac{1}{nk-1} + \frac{1}{n}\right) \\ &> n(k-1)\frac{4}{n(k+1)-1} = \frac{4(k-1)}{k+1-1/n} \\ &> \frac{4(k-1)}{k+1}. \end{aligned}$$

The stated inequality holds for every natural number n if

$$\frac{2(k-1)}{k+1} > 1 + a, \quad \text{i.e.,} \quad k > \frac{3+a}{1-a} \quad (0 \leq a < 1).$$

Reference

Kirov, G.: Problem 3. Matematika (Sofia) 1968, No. 5, 35–36.

3.1.5 For $n = 1, 2, \ldots$, we have

$$\frac{1}{2(2n+1)} < \left| \sum_{k=0}^{n-1} (-1)^k \frac{1}{2k+1} - \frac{\pi}{4} \right| < \frac{1}{2(2n-1)},$$

$$\frac{1}{4(n+1)} < \left| \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{2k} - \frac{1}{2}\log 2 \right| < \frac{1}{4(n-1)}.$$

Reference

Kazarinoff, D. K.: A simple derivation of the Leibniz-Gregory series for $\pi/4$. Amer. Math. Monthly **62**, 726–727 (1955).

3.1.6 Let n, p and q be positive integers and $q > p$. Let

$$s_n(p, q) = \sum_{k=pn+1}^{qn} \frac{1}{k} \quad \text{and} \quad S_n(p, q) = \sum_{k=pn+1}^{qn+1} \frac{1}{k}.$$

D. D. Adamović and M. R. Tasković [1] have proved the following:

1. For any two fixed natural numbers p and $q\ (> p)$, sequence $s_n(p, q)$ is strictly increasing, and therefore,

$$s_1(p, q) \leq s_n(p, q) < \log \frac{q}{p},$$

where $n = 1, 2, \ldots$ The above bounds are the best possible.

2. Let p and q $(> p)$ be fixed natural numbers.

1° If $q \leq \frac{5}{2} p$, and $p \neq 2a + 1$ or $q \neq 5a + b$ $(a = 2, 3, \ldots; b = 1, 2)$, then the sequence $S_n(p, q)$ is monotonely decreasing, and

$$\log \frac{q}{p} < S_n(p, q) \leq S_1(p, q) \qquad (n = 1, 2, \ldots),$$

where the above bounds are the best possible.

2° If $q \geq 3p$, then $S_n(p, q)$ is monotonely increasing. In this case

$$S_1(p, q) \leq S_n(p, q) < \log \frac{q}{p} \qquad (n = 1, 2, \ldots),$$

and those bounds are the best possible.

3° For all other values of p and q, $S_n(p, q)$ is strictly decreasing if n is large enough.

R. P. Lučić and D. Ž. Djoković, independently from the above results, have earlier proved the following:

$$\log \frac{5}{3} < S_n(3, 5) \leq \frac{37}{60}. \tag{1}$$

Putting

$$S_n = \frac{1}{3n + 1} + \frac{1}{3n + 2} + \cdots + \frac{1}{5n} + \frac{1}{5n + 1},$$

we get

$$S_n - S_{n+1} = \frac{1}{3n + 1} + \frac{1}{3n + 2} + \frac{1}{3n + 3} - \frac{1}{5n + 2} - \frac{1}{5n + 3}$$
$$- \frac{1}{5n + 4} - \frac{1}{5n + 5} - \frac{1}{5n + 6}$$
$$= \frac{P_5(n)}{15(3n + 1)(3n + 2)(n + 1)(5n + 2)(5n + 3)(5n + 4)(5n + 6)},$$

where $P_5(n)$ is a polynomial in n of degree 5 with all coefficients positive. Hence, $S_n > S_{n+1}$ for all $n = 1, 2, \ldots$ Hence,

$$\lim_{n \to +\infty} S_n < S_n \leq S_1.$$

It is easy to show that

$$S_1 = \frac{37}{60}, \qquad \lim_{n \to +\infty} S_n = \log \frac{5}{3},$$

which completes the proof of (1).

This is in agreement with the cited results of Adamović and Tasković.

Statement 2° does not give an answer to the question of the best possible bounds of $S_n(p, q)$, for the following cases:

1° $\frac{5}{2} p < q < 3p$,

2° $p = 2a + 1$, $q = 5a + b$ $(a = 2, 3, \ldots; b = 1, 2)$.

However, guided by the statement 3° of 2, and some particular checking, ADAMOVIĆ and TASKOVIĆ have conjectured that $S_n(p, q)$ is strictly decreasing in those cases also.

Remark. The above result is an answer to a problem proposed by D. S. MITRINOVIĆ [2].

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3.1.7 If $n > 1$ is a natural number, then

$$\frac{1}{n} - \frac{1}{2n+1} < \sum_{k=n}^{2n} \frac{1}{k^2} < \frac{1}{n-1} - \frac{1}{2n}.$$

Proof. Since

$$\frac{1}{k(k-1)} > \frac{1}{k^2} > \frac{1}{k(k+1)} \qquad (k > 1),$$

and

$$\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}, \qquad \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

we obtain, for $n > 1$,

$$\frac{1}{n-1} - \frac{1}{2n} = \sum_{k=n}^{2n} \left(\frac{1}{k-1} - \frac{1}{k}\right) > \sum_{k=n}^{2n} \frac{1}{k^2} > \sum_{k=n}^{2n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{n} - \frac{1}{2n+1}.$$

3.1.8 For any positive integer n,

$$\sum_{k=n}^{+\infty} \frac{1}{k^2} < \frac{1}{n - \frac{1}{2}}.$$

Reference

OSTROWSKI **1**, p. 39.

3.1.9 If m and n are natural numbers, then

$$f(m, n) \equiv \frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)} \leq \frac{4}{45}.$$

Proof. We have

$$f(1, 1) = f(1, 2) = f(2, 1) = \frac{1}{12} < \frac{4}{45}.$$

For $k = m + n + 2 \geq 6$, we have

$$f(m, n) \leq \frac{1}{k-1} - \frac{4}{k^2},$$

since

$$\frac{1}{(m+1)(n+1)} \geq \frac{4}{(m+n+2)^2},$$

with equality only if $m = n$.

Since the function $k \mapsto \frac{1}{k-1} - \frac{4}{k^2}$ is decreasing for $k \geq 6$, we have $f(m, n) \leq \frac{4}{45}$.

Remark. This inequality is due to G. Grüss and the above proof can be found in: E. Landau: Über einige Ungleichungen von Herrn G. Grüss. Math. Z. **39**, 742—744 (1935).

3.1.10 Let p and q be positive integers. Let there hold an inequality between p/q and $\sqrt[n+1]{2}$, where n is a positive integer. The sign of this inequality reverses if p/q is replaced by

$$\frac{p^n + p^{n-1}q + \cdots + pq^{n-1} + 2q^n}{p^n + p^{n-1}q + \cdots + pq^{n-1} + q^n},$$

and this expression is a better approximation of $\sqrt[n+1]{2}$ than p/q.

Reference

Késedi, F.: Inequalities (Hungarian). Budapest 1965, pp. 34—35.

3.1.11 If $r_1, \ldots, r_k$ are positive integers, then

$$\sum_{j=1}^{k} \frac{2}{r_j} \prod_{t=1}^{k} \frac{r_t}{r_t + 1} \leq 1,$$

where equality holds if and only if $k \leq 2$ and either r_1 or r_2 is equal to 1.

Reference

Minc, H.: Upper bounds for permanents of (0, 1)-matrices. Bull. Amer. Math. Soc. **69**, 789—791 (1963).

3.1.12 If $n > 1$ is a positive integer, then

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1. \tag{1}$$

Proof. The first inequality in (1) follows from

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sum_{k=1}^{n} \frac{2}{\sqrt{k} + \sqrt{k+1}} = 2\sum_{k=1}^{n} \left(\sqrt{k+1} - \sqrt{k}\right) = 2\sqrt{n+1} - 2.$$

The function $x \mapsto 1/\sqrt{x}$ is strictly decreasing on $[1, n]$. Therefore

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 1 + \sum_{k=2}^{n} \frac{1}{\sqrt{k}} < 1 + \int_1^n \frac{dx}{\sqrt{x}} = 2\sqrt{n} - 1,$$

which we had to prove.

Remark. T. NAGELL and W. LJUNGGREN in Norsk. Mat. Tidsskr. **7**, 106—107 (1925), have proved that

$$2\sqrt{n} - 2 + \frac{1}{\sqrt{n}} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1 \qquad (n > 1),$$

which follows from (1). Indeed,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{n}} + \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}}.$$

Applying the first inequality in (1) with $n - 1$ instead of n, we get

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \frac{1}{\sqrt{n}} + 2\sqrt{n} - 2.$$

3.1.13 Let n be a positive integer. Then

(1) $$(2n-1)^n + (2n)^n < (2n+1)^n \quad \text{for} \quad n > 2;$$

(2) $$(2n+2)^n - (2n)^n < 2(2n+1)^n \sinh \frac{1}{2} \quad \text{for} \quad n \geq 1;$$

(3) $$(2n)^n + (2n+1)^n \geq (2n+2)^n \quad \text{for} \quad 1 \leq n \leq 15.$$

Reference

FRYE, C. M.: Problem E 1624. Amer. Math. Monthly **70**, 891 (1963) and **71**, 683 (1964).

Remark. For the proofs of (2) and (3), see MITRINOVIĆ 2, pp. 236—237.

3.1.14 If $n \geq 2$ is an integer, then

$$n^{n/2} < n! < \left(\frac{n+1}{2}\right)^n.$$

Reference

CAUCHY, A.: Exercices d'analyse, vol. 4. Paris 1847, p. 106.

3.1.15 If $n > 1$ is a natural number, then

$$n^n > (2n-1)!! \quad \text{and} \quad (n+1)^n > (2n)!!.$$

3.1.16 If n is a natural number, then [1]

(1) $$\sqrt{\frac{5/4}{4n+1}} \leq \frac{(2n-1)!!}{(2n)!!} \leq \sqrt{\frac{3/4}{2n+1}}.$$

Remark. For $n = 1, 2, \ldots$ the following WALLIS' inequalities hold

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n}},$$

which are stronger than (1) except for certain first values of n.

D. K. KAZARINOFF [2] demonstrated the following improved inequalities

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}.$$

References

1. RANKIN, R. A.: Introduction to Mathematical Analysis. Oxford-London-New York-Paris **1963**, p. 13.
2. KAZARINOFF, D. K.: On Wallis' formula. Edinburgh Math. Notes **40**, 19–21 (1956).

3.1.17 If $n > 1$ is an integer, then

$$\left(\frac{n(n+1)^3}{8}\right)^n > (n!)^4.$$

3.1.18 If $n \geq 2$ is an integer, then

$$2!\,4!\cdots(2n)! > \big((n+1)!\big)^n.$$

3.1.19 If $n > 2$ is an integer, then

$$n! < 2^{n(n-1)/2}.$$

3.1.20 For arbitrary integer $n \geq 2$ we have

$$\sum_{k=2}^{n} \frac{k^2+1}{k^2-1}\,\frac{1}{k!\,k} < \frac{1}{2}.$$

3.1.21 If n is a natural number ≥ 3, then

$$\frac{2^n}{n} < \frac{n^n}{n!} < 3^{n-1}.$$

3.1.22 Let $n \geq 3$ be a fixed natural number. Then

$$n^n > n!, \quad n^{n^n} > n!!, \ldots,$$

$$n^n < n!!, \quad n^{n^n} < n!!!, \ldots,$$

where $n!! = (n!)!$, $n!!! = ((n!)!)!$, ... and where $n^{n^n} = n^{(n^n)}$, ...

Reference

THORPE, E., and F. RICHMAN: Problem E 1860. Amer. Math. Monthly **74**, 862–863 (1967).

3.1.23 If $x_r = x(x-1)\cdots(x-r+1)$ (r is a natural number) and $x_0 = 1$, then

$$(1) \qquad (2n-1)_{n-1}\,(2n-3)_{n-2}\cdots > n_{n-1}\,n_{n-3}\cdots,$$

where products on the both sides extend over all nonnegative indices of the same parity as $n-1$ and $n > 1$.

A proof of (1), due to S. B. PREŠIĆ, is given in: MITRINOVIĆ **1**, pp. 84–85.

3.1.24 If $n > 1$ is an integer, then

(1) $$n \log n - n < \log n! < \left(n + \frac{1}{2}\right) \log n - n + 1.$$

For the proof of (1) see: MITRINOVIĆ 1, pp. 85–87.

3.1.25 Let r be an integer greater than 1 and put $f(r) = (r!)^{1/r}$. Then

(1) $$1 < \frac{f(r+1)}{f(r)} < 1 + \frac{1}{r}$$

and

$$r \frac{f(r+1)}{f(r)} - (r-1) \frac{f(r)}{f(r-1)} > 1.$$

Inequalities (1) also hold for $r = 1$.

If $r_1, r_2, \ldots, r_s$ are integers greater than 1 and $s \leq r_k$ $(k = 1, \ldots, s)$, then

$$\sum_{k=1}^{s} \frac{1}{f(r_k - 1)} \leq \prod_{k=1}^{s} \frac{f(r_k)}{f(r_k - 1)},$$

with equality if and only if $s = r_1 = \cdots = r_s$.

Reference

MINC, H., and L. SATHRE: Some inequalities involving $(r!)^{1/r}$. Proc. Edinburgh Math. Soc. **(2) 14**, 41–46 (1964/65).

3.1.26 If $n - 1$, p, q are nonnegative integers, with $p + q \leq 10$, then

$$(n-1)^{p+q+1} < \frac{(p+q+1)!}{p!\,q!} \sum_{k=1}^{n-1} k^p (n-k)^q < (n+1)^{p+q+1}.$$

Reference

AGOSTINI, A.: Rileggendo la "Geometria speciosa" di Pietro Mengoli. Periodico Mat. **(4) 20**, 313–327 (1940). — See also Review by J. E. HOFMAN in Jahrbuch über die Fortschritte der Mathematik **66**, 16 (1940).

3.1.27 Let n be a given natural number and let $n_1, \ldots, n_k \in \{0, 1, 2, \ldots\}$, with $n_1 + \cdots + n_k = n$. Then

$$\frac{n_1!\, n_2! \cdots n_k!}{(2n_1)!\,(2n_2)! \cdots (2n_k)!} \leq \frac{1}{2^n}.$$

Reference

KHINTCHINE, A.: Über dyadische Brüche. Math. Z. **18**, 109–116 (1923).

3.1.28 If $n > 2$ is a natural number, then

(1) $$\frac{2^{2n}}{n+1} < \binom{2n}{n} < \frac{(2n+2)^n}{(n+1)!}.$$

If $n \geq 2$ is an integer and if a is a positive number, then

$$\left(\frac{n+a}{a}\right)^n < \binom{r}{n} < \frac{1}{n!}\left(\frac{(n+1)(n+a)}{2a}\right)^n,$$

with $r = \dfrac{n(n+1)}{2a} + n$.

Reference

Bartoš, P., and Š. Znám: On symmetric and cyclic means of positive numbers. Mat.-Fyz. Časopis Sloven. Akad. Vied. **16**, 291–298 (1966).

Remark. For a stronger inequality than the first in (1), see 3.1.29.

3.1.29 For natural numbers $n > 1$ we have

$$\binom{2n}{n} > \frac{4^n}{2\sqrt{n}}.$$

Reference

Sierpiński, W.: Elementary Theory of Numbers. Warszawa 1964, p. 132.

3.1.30 If n and k ($k < n$) are positive integers, then

$$\binom{n}{k} \leq \frac{n^n}{k^k (n-k)^{n-k}}.$$

This result, due to N. Åslund [1], is generalized by G. Kalajdžić in the following form:

Let k be a positive integer and a a real number such that $a > k$. Then

$$\binom{a}{k} \leq \frac{a^a}{bk^k (a-k)^{a-k}}, \quad \text{with} \quad b = (1 + 1/k)^k. \tag{1}$$

G. Kalajdžić proved (1) using induction and monotony of $(1 + 1/k)^k$.

Reference

1. Åslund, N.: The fundamental theorems of information theory II. Nord. Mat. Tidskr. **9**, 105 (1961).

3.1.31 Let $n \geq 13$ be a positive integer and $k = 2, \ldots, n-2$. Then

$$(n+1)^2 < n(n+1-k)\, 2^{k-1},$$

$$n\binom{n}{k} < 2^{1-n+nk-k^2}.$$

Reference

Moon, J. W., and L. Moser: Almost all tournaments are irreducible. Canad. Math. Bull. **5**, 61–65 (1962).

3.1.32 If n and r are natural numbers with $n \geq r > 2$, then

$$\frac{1}{r!} - \frac{1}{2n(r-2)!} < \binom{n}{r} n^{-r} < \frac{1}{r!}.$$

Reference

MACDONALD, J. K. L.: Elementary rigorous treatment of the exponential limit. Amer. Math. Monthly **47**, 157—159 (1940).

3.1.33 Let $r > 0$ and let n be a natural number. Denote

$$I_{r,n} = \sum_{k=0}^{n} \frac{1}{rk+1} \binom{n}{k}.$$

Then

(1) $$\frac{2^{n+1}-1}{n+1} < I_{r,n} < \frac{2^{n+1}-1}{r(n+1)} \quad \text{for} \quad 0 < r < 1 \quad \text{and} \quad n \geq 1;$$

(2) $$I_{1,n} = \frac{2^{n+1}-1}{n+1} \quad \text{for} \quad n \geq 1;$$

(3) $$\frac{2^n}{n} < I_{r,n} < \frac{2^{n+1}-1}{n+1} \quad \text{for} \quad 1 < r \leq 2 \quad \text{and} \quad n \geq 3;$$

(4) $$\frac{2^{n+1}}{r(n+1)} < I_{r,n} < \frac{2^n}{n-1} \quad \text{for} \quad r \geq 2 \quad \text{and} \quad n \geq 2.$$

The second inequality in (3) is valid also for $n = 1$ and $n = 2$. The first inequality in (4) holds also for $n = 1$.

If $r > 0$, then

$$I_{r,n} \geq I_{r,1} = \frac{r+2}{r+1}$$

and this bound is the best possible.

The above results are an answer by M. R. TASKOVIĆ to a problem by D. S. MITRINOVIĆ. See: Problem 94. Mat. Vesnik **6 (21)**, 89—90 (1969).

Concerning the inequalities

$$\frac{2^n}{n} < I_{2,n} < \frac{2^n}{n-1} \quad \text{for} \quad n \geq 3,$$

see H. W. SMITH and J. BARLAZ: Problem 4378. Amer. Math. Monthly **58**, 498—499 (1951).

3.1.34 If x, y, z denote real numbers and n a natural number, such that $x^n + y^n = z^n$, then

$$0 < \binom{z}{n} - \binom{x}{n} - \binom{y}{n} < \binom{z-1}{n-1}.$$

Reference

LEKO, T.: Problem 201. Glasnik Mat.-Fiz. Astronom. Ser. II, Društvo Mat.-Fiz. Hrvatske **12**, 229—231 (1957).

3.1.35 Let n and k $(n > k)$ be natural numbers, and let

$$Q(n,k) = \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k};$$

then

$$Q(n, k)\, e^{\frac{1}{12n+1/4} - \frac{1}{12k} - \frac{1}{12(n-k)}} < \binom{n}{k},$$

and

$$\binom{n}{k} < Q(n, k)\, e^{\frac{1}{12n} - \frac{1}{12k+1/4} - \frac{1}{12(n-k)+1/4}}.$$

Reference

BUCHNER, P.: Bemerkungen zur Stirlingschen Formel. Elem. Math. **6**, 8–11 (1951).

3.1.36 If n denotes a natural number, then

$$\sum_{k=1}^{n} \binom{n}{k}^{1/2} \leq n^{1/2}(2^n - 1)^{1/2}.$$

3.1.37 If m and n are nonnegative integers and $a \geq 0$, then

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+a}{m-k}\binom{n-k+a}{n-k}\binom{k-a-2}{k} \geq 0.$$

Reference

LORENTZ, G. G., and K. ZELLER: Abschnittslimitierbarkeit und der Satz von Hardy-Bohr. Arch. Math. (Basel) **15**, 208–213 (1964).

3.1.38 If m and n are positive integers, then

$$2^{n-1}\binom{m+n-1}{n-1} < \sum_{k=0}^{n-1} 2^k \binom{m+k}{k} < 2^n \binom{m+n-1}{n-1}.$$

Reference

MAKAI, E.: The first main theorem of P. Turán. Acta Math. Acad. Sci. Hung. **10**, 405–411 (1959).

3.1.39 Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be positive integers such that $b_i \leq a_i$ $(i = 1, \ldots, m)$. Then

$$0 < \prod_{i=1}^{m} \binom{a_i}{b_i} \leq \binom{A}{B}, \tag{1}$$

where $A = \sum_{i=1}^{m} a_i$ and $B = \sum_{i=1}^{m} b_i$.

Proof. By comparing the coefficients of x^B in the expansion

$$(1+x)^A = \prod_{i=1}^{m} (1+x)^{a_i} = \prod_{i=1}^{m} \left[1 + \binom{a_i}{1} x + \cdots + \binom{a_i}{b_i} x^{b_i} + \cdots + x^{a_i}\right],$$

we conclude that (1) is true.

Remark. A. Moór gave, without proof, the generalized inequalities

$$0 < \prod_{i=1}^{m} \binom{p_i}{q_i} \binom{a_i}{b_i} \leq \binom{A^*}{B^*}, \qquad 0 \leq \prod_{i=1}^{m} \binom{a_i}{b_i} \leq \binom{A^r}{B^r}^{1/r},$$

where r, p_i, q_i $(i = 1, \ldots, m)$, $A^* = \sum_{i=1}^{m} p_i a_i$, $B^* = \sum_{i=1}^{m} q_i b_i$ are positive integers, A and B being the numbers defined above.

References

Bareiss, E., and F. Goldner: Problem 132. Elem. Math. 7, 117 (1952).
Moór, A.: Problem 132. Elem. Math. 7, 117 (1952).

3.2 Inequalities Involving Algebraic Functions

3.2.1 If $a \neq 1$ is real, then

$$(1 + a + a^2)^2 < 3(1 + a^2 + a^4).$$

3.2.2 If a, b, c are real numbers, then

$$(a + b + c + x)^2 > 8(ac + bx)$$

for all x, if and only if $a > b > c$, or $a < b < c$.

3.2.3 For every natural number n and $0 \leq x \leq 1$,

$$x^{n-1}(1 - x)^n \leq \frac{1}{2^{2n-2}}.$$

Reference

Ostrowski 3, p. 95.

3.2.4 If n is a natural number and $a \geq 0$, then

$$a + a^2 + \cdots + a^{2n} \leq n(a^{2n+1} + 1).$$

3.2.5 If $0 \leq x \leq 1$, then

$$\prod_{k=0}^{n} \left(1 - \frac{1}{2} x^{2^k}\right) \geq 1 - x + \frac{x}{2^{n+1}}.$$

3.2.6 For any natural numbers n and p and $0 \leq x \leq n$, we have

$$|x^{p-1}(x - 1)^p (x - 2)^p \cdots (x - n)^p| \leq (n!)^p.$$

Reference

Ostrowski 3, p. 95.

3.2.7 If

(1) $$y = \frac{x^2 - 2mx + p^2}{x^2 + 2mx + p^2},$$

then, for all real x,

(2) $\min\left(\frac{p-m}{p+m}, \frac{p+m}{p-m}\right) \leq y \leq \max\left(\frac{p-m}{p+m}, \frac{p+m}{p-m}\right) \quad (0 < |m| < |p|)$,

(3) $y \leq \min\left(\frac{p-m}{p+m}, \frac{p+m}{p-m}\right)$, or $y \geq \max\left(\frac{p-m}{p+m}, \frac{p+m}{p-m}\right) \quad (0 < |p| < |m|)$.

Proof. If $m = 0$, then $y = 1$. Suppose that $m \neq 0$ and write (1) in the form

$$(y-1)\, x^2 + 2m(y+1)\, x + p^2(y-1) = 0.$$

A necessary and sufficient condition that the roots of this equation in x be real is

$$(m^2 - p^2)\, y^2 + 2(m^2 + p^2)\, y + m^2 - p^2 \geq 0.$$

We suppose also that $|m| \neq |p|$. First, we have

$$(m^2 - p^2)\, y^2 + 2(m^2 + p^2)\, y + m^2 - p^2 = (m^2 - p^2)\,(y - y_1)\,(y - y_2),$$

where

$$y_1 = \frac{p-m}{p+m}, \qquad y_2 = \frac{p+m}{p-m}.$$

If $mp > 0$ and $0 < |m| < |p|$, then

$$\frac{p-m}{p+m} \leq y \leq \frac{p+m}{p-m}.$$

If $mp < 0$ and $0 < |m| < |p|$, then

$$\frac{p+m}{p-m} \leq y \leq \frac{p-m}{p+m}.$$

If $mp > 0$ and $0 < |p| < |m|$, then

$$y \leq \frac{p+m}{p-m} \quad \text{or} \quad y \geq \frac{p-m}{p+m}.$$

If $mp < 0$ and $0 < |p| < |m|$, then

$$y \leq \frac{p-m}{p+m} \quad \text{or} \quad y \geq \frac{p+m}{p-m}.$$

This proves (2) and (3).

3.2.8 If $a = \cos\alpha$, $c = \sin\alpha$, $b^2 = \sin 2\alpha$ $(0 < \alpha < \pi/4)$, then

$$(\sec\alpha - 1)\,(\operatorname{cosec}\alpha + 1) \leq \frac{ax^2 + bx + c}{cx^2 + bx + a} \leq (\sec\alpha + 1)\,(\operatorname{cosec}\alpha - 1).$$

3.2.9 If $\theta \neq k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$, then, for all real x,

$$-1 \leq \frac{x^2\cos\theta - 2x + \cos\theta}{x^2 - 2x\cos\theta + 1} \leq +1.$$

3.2.10 Let $a \pm b \neq 2n\pi$ (n integer) and

$$y = \frac{x^2 - 2x \cos a + 1}{x^2 - 2x \cos b + 1}.$$

If $\sin b = 0$ ($b = k\pi$), then

$$\frac{1}{2}[1 + (-1)^k \cos a] \leq y < +\infty.$$

If $\sin b \neq 0$, then

$$\min\left(\frac{1 - \cos a}{1 - \cos b}, \frac{1 + \cos a}{1 + \cos b}\right) \leq y \leq \max\left(\frac{1 - \cos a}{1 - \cos b}, \frac{1 + \cos a}{1 + \cos b}\right).$$

3.2.11 Let $x > 0$, $x \neq 1$ and let n be a positive integer. Then

$$x + \frac{1}{x^n} > 2n \frac{x - 1}{x^n - 1}, \tag{1}$$

Proof. We can transform (1) into

$$\frac{(x^{n+1} + 1)(x^n - 1)}{x - 1} > 2nx^n. \tag{2}$$

The identity

$$\frac{(x^{n+1} + 1)(x^n - 1)}{x - 1} = (x^{n+1} + 1)(x^{n-1} + x^{n-2} + \cdots + 1)$$
$$= x^n \sum_{k=1}^{n} \left(x^k + \frac{1}{x^k}\right)$$

together with $x^k + \frac{1}{x^k} > 2$, implies (2).

Remark. This proof is due to R. Ž. Djordjević.

Reference

Beck, I.: Problem 224. Nord. Mat. Tidskr. **10**, 96 (1962).

3.2.12 If n is natural number and $0 < x \leq 1$, then

$$1 \leq \frac{1 + nx^{n+1}}{(n + 1)x^n} \leq 1 + \frac{n}{2x^n}(1 - x)^2.$$

This result is due to V. I. Levin.

3.2.13 For all integral values of n and r with $n \geq 1$, $0 \leq r \leq n$, and for all values of x with $0 < x < 1$, the following inequality holds:

$$\binom{n}{r} x^r (1 - x)^{n-r} < \frac{1}{2enx(1 - x)^{1/2}}.$$

Reference

Herzog, F.: Problem 4186. Amer. Math. Monthly **54**, 485–486 (1947).

3.2.14 If $x \geq 0$, then

(1) $$\sqrt{1+x} \geq 1 + \frac{x}{2} - \frac{x^2}{8}.$$

If $x \geq -1$, then

(2) $$\sqrt{1+x} \leq 1 + \frac{x}{2}.$$

Proof. By TAYLOR's formula

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8}(1+\theta x)^{-3/2} \qquad (0 < \theta < 1),$$

so it follows immediately that (1) and (2) hold.

3.2.15 If $x > 0$, then

$$0 < \sqrt[3]{1+x} - 1 - \frac{1}{3}x + \frac{1}{9}x^2 < \frac{5}{81}x^3.$$

3.2.16 If $-1 < x < 1$ and n is a natural number, then

$$1 + \frac{x}{n} - \frac{n-1}{2n^2}x^2 \leq \sqrt[n]{1+x} \leq 1 + \frac{x}{n}.$$

3.2.17 If $n \geq 3$ is an integer and $x > -1/2$, then

$$x^n + (1+2x)^{n/2} \leq (x+1)^n.$$

3.2.18 For any real numbers a, b, c,

$$(bc + ca + ab)^2 \geq 3abc(a+b+c).$$

3.2.19 If $a, b, c > 0$ and $a \neq b \neq c \neq a$, then

$$a^4 + b^4 + c^4 > abc(a+b+c).$$

3.2.20 Let x, y, z, u, v, w be real numbers. Then

$$x^2 + y^2 + z^2 - yz - zx - xy + u^2 + v^2 + w^2$$
$$- vw - wu - uv \geq \sqrt{3}\begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}.$$

Proof. Let (u, x), (v, y) and (w, z) be three points determining a triangle whose sides will be denoted by a, b and c and area by P. Then, the above inequality is equivalent to

(1) $$\frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) \geq 3P.$$

Of the triangles having a given perimeter $a + b + c = 3p$, the equilateral triangle has the maximum area $\frac{\sqrt{3}}{4} p^2$. Also, $a^2 + b^2 + c^2 > 3p^2$, unless $a = b = c = p$, in which case the two members are equal.

Hence, equality holds if the triangle ABC is equilateral.

Remark. Concerning (1) see 4.4 in book [17] cited at the end of Preface.

Reference

LANGMAN, H., and F. AYRES: Problem 3315. Amer. Math. Monthly **36**, 238 (1929).

3.2.21 If $a_{n-1} + a_{n+1} \geq 2a_n$ for $n = 2, 3, \ldots$, then

$$A_{n-1} + A_{n+1} \geq 2A_n,$$

for $n = 2, 3, \ldots$, where

$$A_n = \frac{a_1 + \cdots + a_n}{n}.$$

Reference

OZEKI, N.: On some inequalities (Japanese). J. College Arts Sci. Chiba Univ. **4**, No. 3, 211–214 (1965).

3.2.22 If $a_1 > \cdots > a_n > 0$, then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} (3k^2 + k)\, a_k\right) - 4\left(\sum_{k=1}^{n} k a_k\right)^2 > 0,$$

and

$$5\left(\sum_{k=1}^{n} k a_k\right)^2 - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} k a_k\right) - 3\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} k^2 a_k\right) > 0.$$

The above inequalities improve those of G. MANTELLINO.

Reference

LOVERA, P.: Sopra alcune diseguaglianze che si presentano nella matematica attuariale. Giorn. Ist. Ital. Attuari **19**, 131–139 (1956).

3.2.23 If a, b, c are real numbers satisfying $a^3 + b^3 + c^3 = 0$, then

$$(1) \qquad (\Sigma a^2)^3 \leq \left(\Sigma (b - c)^2\right) \Sigma a^4.$$

Proof. If $s_n = \Sigma a^n$, then

$$\Delta = (a-b)^2 (b-c)^2 (c-a)^2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 = \begin{vmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

$$= s_4 (3s_2 - s_1^2) - s_3 (3s_3 - s_1 s_2) + s_2 (s_1 s_3 - s_2^2).$$

Using

$$\Sigma (a - b)^2 = 2s_2 - 2\Sigma ab = 3s_2 - s_1^2,$$

we get

$$\left(\Sigma (a-b)^2\right) \Sigma a^4 - (\Sigma a^2)^3 = s_4(3s_2 - s_1^2) - s_2^3$$
$$= \Delta + s_3(3s_3 - 2s_1 s_2).$$

Since $s_3 = 0$, we infer that

$$\left(\Sigma (a-b)^2\right) \Sigma a^4 - (\Sigma a^2)^3 = \Delta \geq 0,$$

which proves (1).

Reference

LYNESS, R. C., and L. CARLITZ: Problem 4791. Amer. Math. Monthly **66**, **241** (1959).

3.2.24 For all real numbers a_i, b_i, c_i $(i = 1, 2)$,

(1) $(a_1b_2 + a_2b_1 + b_1c_2 + b_2c_1 + c_1a_2 + c_2a_1 - 2a_1a_2 - 2b_1b_2 - 2c_1c_2)^2$
$\leq 4(a_1^2 + b_1^2 + c_1^2 - b_1c_1 - c_1a_1 - a_1b_1)(a_2^2 + b_2^2 + c_2^2 - b_2c_2 - c_2a_2 - a_2b_2)$,

with equality holding if and only if $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$

Proof. The inequality

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0,$$

which is valid for all real a, b, c, is equivalent to

$$a^2 + b^2 + c^2 \geq bc + ca + ab, \tag{2}$$

with equality if and only if $a = b = c$.

Setting $a = a_1 + ta_2$, $b = b_1 + tb_2$, $c = c_1 + tc_2$ (t real) in (2), we get

$$(a_2^2 + b_2^2 + c_2^2 - b_2c_2 - c_2a_2 - a_2b_2)\, t^2$$
$$- (a_1b_2 + a_2b_1 + b_1c_2 + b_2c_1 + c_1a_2 + c_2a_1 - 2a_1a_2 - 2b_1b_2 - 2c_1c_2)\, t$$
$$+ (a_1^2 + b_1^2 + c_1^2 - b_1c_1 - c_1a_1 - a_1b_1) \geq 0.$$

Since this inequality is valid for all real t, the discriminant of the quadratic expression in t on the left side of the above inequality must be nonpositive. This condition yields (1).

Reference

COŞNIŢĂ, C., and F. TURŢOIU: Collection of Problems in Algebra (Romanian). Bucuresti 1965, pp. 172–173.

Generalization due to P. R. BEESACK. Beginning with

$$(n-1)\sum_{i=1}^{n} a_i^2 = \sum_{1\leq i<j\leq n} (a_i^2 + a_j^2) \geq 2 \sum_{1\leq i<j\leq n} a_i a_j$$

for all real a_i, by setting $a_i = b_i + \tau c_i$ and using the discriminant, one can obtain

$$\left[(n-1)\sum_{i=1}^{n} b_i c_i - \sum_{1\le i<j\le n} (b_i c_j + c_i b_j)\right]^2$$
$$\le \left[(n-1)\sum_{i=1}^{n} b_i^2 - 2\sum_{i<j} b_i b_j\right]\left[(n-1)\sum_{i=1}^{n} c_i^2 - 2\sum_{i<j} c_i c_j\right]$$

valid for all real b_i, c_i.

3.2.25 Let $a = (a_1, \ldots, a_n)$ be a sequence of real numbers and $b = (b_1, \ldots, b_n)$ be a sequence of positive numbers. Then

$$\min_{1\le k\le n} \frac{a_k}{b_k} \le \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} \le \max_{1\le k\le n} \frac{a_k}{b_k}. \tag{1}$$

Equality holds in both above inequalities if and only if the sequences a and b are proportional.

These inequalities are called CAUCHY's.

Proof. If $m = \min_k \frac{a_k}{b_k}$ and $M = \max_k \frac{a_k}{b_k}$, then we have successively

$$m \le \frac{a_k}{b_k} \le M,$$
$$m b_k \le a_k \le M b_k,$$
$$m \sum_{k=1}^{n} b_k \le \sum_{k=1}^{n} a_k \le M \sum_{k=1}^{n} b_k,$$
$$m \le \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} \le M,$$

i.e., (1).

Application. If $x \ge 0$, then

$$\frac{1}{n} \le \frac{1 + 2x + \cdots + nx^{n-1}}{n + (n-1)x + \cdots + x^{n-1}} \le n,$$

and

$$\frac{1}{n} \le \frac{1 + 2x + \cdots + nx^{n-1}}{1 + 2^2 x + \cdots + n^2 x^{n-1}} \le 1.$$

All the bounds in the above inequalities are the best possible.

Reference

CAUCHY, A.: Oeuvres complètes II, vol. 3. Paris 1897, pp. 368–369.

3.2.26 Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers such that

$$a \leq a_i \leq A \quad \text{and} \quad b \leq b_i \leq B \qquad (i = 1, \ldots, n).$$

Then

$$-(A-a)(B-b)\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right) \leq \sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$$
$$\leq (A-a)(B-b)\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right).$$

Reference

BIERNACKI, M., H. PIDEK and C. RYLL-NARDZEWSKI: Sur une inégalité entre des intégrales définies. Ann. Univ. Mariae Curie-Skłodowska **A 4**, 1–4 (1950).

3.2.27 If the real sequence $a = (a_1, a_2, \ldots, a_{2n+1})$ is convex, then

$$\frac{a_1 + a_3 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \cdots + a_{2n}}{n}, \tag{1}$$

with equality holding if and only if a presents an arithmetic progression.

Proof. Since a is convex, we have

$$a_k - 2a_{k+1} + a_{k+2} \geq 0, \qquad \text{for } k = 1, 2, \ldots, 2n-1. \tag{2}$$

In virtue of this, we conclude that for $k = 1, \ldots, n$,

$$k(n-k+1)(a_{2k-1} - 2a_{2k} + a_{2k+1}) \geq 0$$

and

$$k(n-k)(a_{2k} - 2a_{2k+1} + a_{2k+2}) \geq 0.$$

Adding these inequalities, we get (1).

In order that equality holds in (1), equality should hold in each of inequalities (2), which will happen if and only if the sequence a is an arithmetic progression.

Remark. (1) was proved by E. J. NANSON [1]. For $a_k = x^{k-1}$ $(k = 1, 2, \ldots, 2n+1)$, where x is a positive number, (1) becomes

$$\frac{1 + x^2 + x^4 + \cdots + x^{2n}}{x + x^3 + \cdots + x^{2n-1}} \geq \frac{n+1}{n}, \tag{3}$$

with equality if and only if $x = 1$. For $x \neq 1$, (3) is equivalent to

$$\frac{x^{2n+2} - 1}{x(x^{2n} - 1)} > \frac{n+1}{n}. \tag{4}$$

As far as we know, inequalities (3) and (4) were first proposed as a problem by J. M. WILSON [2]. After this, a number of proofs of (3) and (4) were given (see [2], [3] and [4]), in spite of which, 90 years later they were again proposed in [5]. However, sharper inequalities than (3) and (4) were also known from 1868: C. TAYLOR [6] proved that, for $x \neq 1$ and $x > 0$,

(5) $$\frac{1 + x^2 + x^4 + \cdots + x^{2n}}{x + x^3 + \cdots + x^{2n-1}} > \frac{n+1}{n} + \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$$

and

(6) $$\frac{1 + x^2 + x^4 + \cdots + x^{2n}}{x + x^3 + \cdots + x^{2n-1}} > \frac{1}{2}\,\frac{n+1}{n}\left(x + \frac{1}{x}\right).$$

It would be interesting to try to obtain for the sequence a which appears in (1) inequalities analogous to (5) and (6).

References

1. NANSON, E. J.: An inequality. Messenger Math. **34**, 89—90 (1904).
2. WILSON, J. M.: Problem 1521. Educ. Times **3**, 14—15 (1865); **5**, 98—99 (1866); **6**, 105 (1866).
3. INGLEBY, C. M.: An inequality. Oxford, Cambridge, Dublin Messenger Math. **3**, 235 (1866).
4. SHARPE, J.: An inequality. Oxford, Cambridge, Dublin Messenger Math. **4**, 123—125 (1868).
5. OLDS, C. D.: Problem E 1167. Amer. Math. Monthly **63**, 43 (1956).
6. TAYLOR, C.: Problem 2426. Educ. Times **8**, 58—59 (1868), and **10**, 31—32 (1868).

3.2.28 If $x_i \leq x_k$ $(i < k)$ and $y_i \leq y_k$ $(i < k)$, or if $x_i \geq x_k$ $(i < k)$ and $y_i \geq y_k$ $(i < k)$, then

$$D_n \leq D_{n+1} \text{ with } D_r = r\sum_{v=1}^{r} x_v y_v - \left(\sum_{v=1}^{r} x_v\right)\left(\sum_{v=1}^{r} y_v\right).$$

Proof. We have

$$\begin{aligned} D_{n+1} - D_n &= \sum_{v=1}^{n} x_v y_v + n x_{n+1} y_{n+1} - y_{n+1}\sum_{v=1}^{n} x_v - x_{n+1}\sum_{v=1}^{n} y_v \\ &= \sum_{v=1}^{n} x_v (y_v - y_{n+1}) + x_{n+1}\sum_{v=1}^{n} (y_{n+1} - y_v) \\ &= \sum_{v=1}^{n} (x_{n+1} - x_v)(y_{n+1} - y_v) \geq 0. \end{aligned}$$

Equality holds if and only if $x_i = x_{n+1}$ for $i \in I \subset \{1, \ldots, n\}$ and $y_i = y_{n+1}$ for $i \in \{1, \ldots, n\} \setminus I$.

Remark. If $x_i \leq x_k (i < k)$ and $y_i \geq y_k$ $(i < k)$, or if $x_i \geq x_k$ $(i < k)$ and $y_i \leq y_k$ $(i < k)$, then the reversed inequality holds.

This proof is due to R. R. JANIĆ.

3.2.29 If $a_k > 0$ $(k = 1, \ldots, n)$, then

(1) $$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq n^2.$$

Proof. Putting $\alpha_k = \sqrt{a_k}$, $\beta_k = 1/\sqrt{a_k}$ in CAUCHY's inequality

$$\left(\sum_{k=1}^{n} \alpha_k^2\right)\left(\sum_{k=1}^{n} \beta_k^2\right) \geq \left(\sum_{k=1}^{n} \alpha_k \beta_k\right)^2,$$

we get

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \left(\sum_{k=1}^{n} \sqrt{a_k}\,\frac{1}{\sqrt{a_k}}\right)^2 = \left(\sum_{k=1}^{n} 1\right)^2 = n^2.$$

Equality holds in (1) if and only if all the a_k's are equal.

Remark. This presents an other proof of the inequality $A_n(a) \geq H_n(a)$.

3.2.30 For any real numbers a_k we have

$$\left(\sum_{k=1}^{n} \frac{a_k}{k}\right)^2 \leq \left(\sum_{k=1}^{n} k^3 a_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{k^5}\right).$$

3.2.31 If a_k, b_k and c_k $(k = 1, \ldots, n)$ are real numbers, then

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right)\left(\sum_{k=1}^{n} c_k^2\right)^2.$$

Proof. Applying CAUCHY's inequality twice, we get

$$\left(\sum_{k=1}^{n} (a_k b_k)\, c_k\right)^2 \leq \left(\sum_{k=1}^{n} (a_k b_k)^2\right)\left(\sum_{k=1}^{n} c_k^2\right),$$

$$\left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^2 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right).$$

Therefore,

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^4\right)^{1/2}\left(\sum_{k=1}^{n} b_k^4\right)^{1/2}\left(\sum_{k=1}^{n} c_k^2\right),$$

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right)\left(\sum_{k=1}^{n} c_k^2\right)^2,$$

which we had to prove.

3.2.32 For any real numbers a_k and b_k we have

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right).$$

3.2.33 For all natural numbers n and all real numbers $a > 1$,

$$\sum_{i=0}^{n} \frac{a^{ij}}{i!\,j!}\, a^{i-j} < a \sum_{i=0}^{n} \frac{a^{ij}}{i!\,j!},$$

where $i + j = n$.

Reference

Lewin, M.: An inequality and its application to a property of Mahler's partition functions. J. London Math. Soc. **43**, 429–432 (1968).

3.2.34 If $a_k > 0$ $(k = 1, \ldots, n)$ and $\prod_{k=1}^{n} a_k = b^n$, then

$$(1) \qquad \prod_{k=1}^{n} (1 + a_k) \geq (1 + b)^n.$$

Proof. Since $f(x) = \log(1 + e^x)$ is convex on $(-\infty, +\infty)$, we have

$$\sum_{k=1}^{n} \log(1 + e^{x_k}) \geq n \log\left(1 + \exp\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right).$$

Replacing x_k by $\log a_k$ $(k = 1, \ldots, n)$, we get

$$\log \prod_{k=1}^{n} (1 + a_k) \geq n \log (1 + b),$$

which is equivalent to (1).

Equality holds in (1) if and only if all the a_k's are equal.

3.2.35 In the case when all the factors are positive, inequality

$$a_1a_2a_3 \geq \prod (a_2 + a_3 - a_1) = (a_2 + a_3 - a_1)(a_3 + a_1 - a_2)(a_1 + a_2 - a_3)$$

holds.

If all the factors are positive, inequalities

$$a_1a_2a_3a_4 \geq \prod (a_2 + a_3 + a_4 - 2a_1),$$
$$\vdots$$
$$a_1 \cdots a_n \geq \prod (a_2 + \cdots + a_n - (n - 2)\, a_1)$$

also hold.

More precisely, the following can be stated in connection with the inequality

$$(1) \qquad a_1 \cdots a_n \geq \prod (a_2 + \cdots + a_n - (n - 2)\, a_1),$$

where $a_1, \ldots, a_n > 0$ for $n = 3, 4, \ldots$:

1° If all the factors are nonnegative, (1) holds, and in this case reduces to equality if and only if $a_1 = a_2 = \cdots = a_n$.

2° For $n = 3$, (1) is valid without any restrictions, and reduces to an equality if and only if $a_1 = a_2 = a_3$.

3° For each $n = 4, 5, \ldots$ there exist (positive) a_i's so that (1) does not hold.

Proof. Putting $a_{n+m} = a_m$ $(m = 1, \ldots, n-1)$ and

(2) $$a_{k+1} + a_{k+2} + \cdots + a_{k+n-1} - (n-2)\, a_k = x_k \quad (k = 1, \ldots, n),$$

and assuming

(3) $$x_k \geq 0 \qquad (k = 1, \ldots, n),$$

it is easy to check that (2) implies

(4) $$(n-1)\, a_m = \sum_{r=1}^{n-1} x_{m+r} \qquad (m = 1, \ldots, n),$$

with $x_{n+m} = x_m$ $(m = 1, \ldots, n-1)$. Using (3) and the arithmetic-geometric mean inequality, we obtain

(5) $$\prod_{m=1}^{n} a_m = \prod_{m=1}^{n} \left(\frac{1}{n-1} \sum_{r=1}^{n-1} x_{m+r}\right) \geq \prod_{m=1}^{n} \left(\prod_{r=1}^{n-1} x_{m+r}\right)^{\frac{1}{n-1}}$$
$$= \prod_{m=1}^{n} x_m = \prod \, (a_2 + a_3 + \cdots + a_n - (n-2)\, a_1).$$

Under condition (3), equality in (5) holds if and only if $x_1 = x_k$ $(k = 2, \ldots, n)$, i.e., according to (2) and (4) if and only if $a_1 = a_k$ $(k = 2, \ldots, n)$.

Let $n = 3$. The following inequalities

$$x_k < 0, \quad x_l < 0 \qquad (1 \leq k \leq l \leq 3),$$

with the notations already introduced, imply the inequality $a_m < 0$ $(m \in \{1, 2, 3\},\ m \neq k, l)$, contrary to the hypothesis $a_1, a_2, a_3 > 0$. Therefore, at most one factor on the right side of (1) may be negative which implies both assertions in 2°.

Let $n \geq 4$. Suppose that (1) holds without any restrictions on $a_1, \ldots, a_n > 0$. Set $a_1 = a_2 = 1$, $a_3 = a_4 = \cdots = a_n = \varepsilon > 0$. Then

$$\varepsilon^{n-2} \geq (2-\varepsilon)^{n-2} \left(-n + 3 + (n-2)\,\varepsilon\right)^2 \qquad (\varepsilon > 0)$$

and when $\varepsilon \to 0$,

$$0 \geq 2^{n-2}\, (n-3)^2 > 0,$$

which is impossible. Thus, assertion 3° is proved.

The above result, due to D. D. Adamović, is an answer to Problem 46 proposed by D. S. Mitrinović in Mat. Vesnik **3 (18)**, 218–220 (1966).

3.2.36 Let $a_1, \ldots, a_n$ be positive numbers such that $\sum_{i=1}^{n} a_i = 1$. Then

$$\sum_{i<k} \frac{a_i a_k}{a_i + a_k} \leq \frac{n-1}{4}, \quad \sum \frac{a_1}{a_2 \cdots a_n} \geq n^{n-1}, \quad \sum \frac{a_1 + \cdots + a_k}{a_1 \cdots a_k} \geq k \binom{n}{k} n^{k-1}.$$

Equality in these inequalities holds if $a_1 = \cdots = a_n = 1/n$.

Reference

BAÏČEV, I., and M. PETKOV: Problem 48. Fiz.-Mat. Spis., Sofia **5**, **142–143** (1962).

3.2.37 Let $a_k \in (0, 1)$ for $k = 1, \ldots, n$. Then

$$\prod_{k=1}^{n} (1 - a_k) > 1 - \sum_{k=1}^{n} a_k, \tag{1}$$

$$\prod_{k=1}^{n} (1 + a_k) < \frac{1}{\prod_{k=1}^{n} (1 - a_k)}. \tag{2}$$

If $a_k \in (0, 1)$ for $k = 1, \ldots, n$ and $\sum_{k=1}^{n} a_k < 1$, then

$$\prod_{k=1}^{n} (1 + a_k) < \frac{1}{1 - \sum_{k=1}^{n} a_k}, \tag{3}$$

$$\prod_{k=1}^{n} (1 - a_k) < \frac{1}{1 + \sum_{k=1}^{n} a_k}. \tag{4}$$

Combining the above inequalities (1), (2), (3), (4), with the same conditions on the a_k's, we have

$$\frac{1}{1 - \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n} (1 + a_k) > 1 + \sum_{k=1}^{n} a_k,$$

$$\frac{1}{1 + \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n} (1 - a_k) > 1 - \sum_{k=1}^{n} a_k.$$

All the above inequalities are usually called WEIERSTRASS' inequalities.

Reference

BROMWICH, T. J. I'A: An Introduction to Theory of Series. 2nd ed., London 1955, pp. 104–105.

3.2.38 If, for $n > 2$,

$$\sum_{i=1}^{n} x_i = p, \quad \sum_{i<k} x_i x_k = q, \tag{1}$$

then

$$(2) \qquad \frac{p}{n} - \frac{n-1}{n}\sqrt{p^2 - \frac{2n}{n-1}q} < x_i < \frac{p}{n} + \frac{n-1}{n}\sqrt{p^2 - \frac{2n}{n-1}q}$$

for $i = 1, \ldots, n$.

Proof. Starting with the obvious inequality $\sum_{i<k} (x_i - x_k)^2 \geq 0$, we get

$$(3) \qquad (n-1)\sum_{i=1}^{n} x_i^2 - 2q \geq 0.$$

Since, by (1), $\sum_{i=1}^{n} x_i^2 = p^2 - 2q$, inequality (3) becomes

$$(4) \qquad p^2 - \frac{2n}{n-1} q \geq 0.$$

Writing equalities (1) in the form

$$(5) \qquad x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n = p - x_i,$$

$$(6) \qquad x_1x_2 + \cdots + x_1x_{i-1} + x_1x_{i+1} + \cdots + x_{n-1}x_n$$
$$= q - x_i(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n) = q - x_i(p - x_i),$$

then for equalities (5) and (6), inequality (4) also holds with p, q, n replaced by $p - x_i$, $q - x_i(p - x_i)$, $n - 1$, respectively. So we get

$$(p - x_i)^2 - \frac{2(n-1)}{n-2}(q - px_i + x_i^2) \geq 0,$$

i.e.,

$$(7) \qquad nx_i^2 - 2px_i + 2(n-1)q - p^2(n-2) \leq 0.$$

The discriminant of the quadratic polynomial in (7)

$$(n-1)^2\left(p^2 - \frac{2n}{n-1}q\right)$$

is nonnegative according to (4).

Since all the x_i's are real, (7) implies (2).

Remark. This result is due to E. LAGUERRE. See also B. S. MADHAVA RAO and B. S. SASTRY: On the limits for the roots of a polynomial equation. J. Mysore Univ. 1, 5–8 (1940).

3.2.39 If c_1, c_2, c_3 are elementary symmetric functions of n real numbers $x_1, \ldots, x_n$ of degrees 1, 2, 3 respectively, then

$$\frac{(3c_3 - c_1c_2)^2}{2(c_1^2 - 2c_2)(2c_2^2 - 3c_1c_3)} \leq 1 - \frac{1}{n}.$$

Equality occurs if and only if all the $x_1, \ldots, x_n$ are equal.

This inequality is equivalent to

$$2(n-1)(\Sigma x_1^2)(2\Sigma x_1^2 x_2^2 + \Sigma x_1^2 x_2 x_3) - n(\Sigma x_1^2 x_2)^2 \geq 0.$$

Reference

OBRECHKOFF, N.: Sur le théorème de Hermite et Poulain. C. R. Acad. Sci. Paris **249**, 21–22 (1959).

3.2.40 Let $a = (a_1, \ldots, a_n)$ be a sequence of positive numbers and

$$P_n(a) = \sum_{k=1}^{n} \frac{1}{1+a_k}, \qquad Q_n(a) = \frac{n}{1+G_n(a)},$$

where $G_n(a)$ is geometric mean of $a_1, \ldots, a_n$.

If $a_1 \cdots a_{n-1} > 1$ and $a_n \geq (a_1 \cdots a_{n-1})^{\frac{-1}{n+1}}$, then

$$P_n(a) - Q_n(a) \geq P_{n-1}(a) - Q_{n-1}(a). \tag{1}$$

If $a_1 \cdots a_{n-1} < 1$ and $a_n \leq (a_1 \cdots a_{n-1})^{\frac{-1}{n+1}}$, the above inequality is reversed.

By a repeated use of inequality (1) the following result can be obtained:

If $\prod_{i=1}^{k} a_i \geq 1$ and $a_{k+1} \geq (a_1 \cdots a_k)^{-\frac{1}{k+2}}$ $(k = 1, \ldots, n-1)$, then

$$P_n(a) \geq Q_n(a). \tag{2}$$

If $\prod_{i=1}^{k} a_i \leq 1$ and $a_{k+1} \leq (a_1 \cdots a_k)^{-\frac{1}{k+2}}$ $(k = 1, \ldots, n-1)$, the above inequality is reversed.

Remark. The inequalities (1) and (2) were proved by D. S. MITRINOVIĆ and P. M. VASIĆ [1].

If $a_1, \ldots, a_n > 1$ (or $a_1, \ldots, a_n < 1$) P. HENRICI's inequality proved in [2] can be obtained from the inequality (2).

For a generalization of (1) and (2) see the paper [3] of P. S. BULLEN.

References

1. MITRINOVIĆ, D. S., and P. M. VASIĆ: Généralisation d'une inégalité de Henrici. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **210–228**, 35–38 (1968).
2. HENRICI, P.: Problem **245**. Elem. Math. **11**, 112 (1956).
3. BULLEN, P. S.: An inequality due to Henrici. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **247–273**, 21–26 (1969).

3.2.41 If a_i $(i = 1, \ldots, n)$ are positive numbers such that $a_1 < \cdots < a_n \leq 1$, then

$$\frac{A_n(a)}{A_n(a) + G_n(a)^n} \geq \frac{1}{n} P_n(a), \tag{1}$$

where A_n and G_n are arithmetic and geometric means and $P_n(a)$ has the same meaning as in 3.2.40.

Proof. Using the inequalities

$$A_n(a) \geq G_n(a) \quad \text{and} \quad G_n(a)^{n-1} \leq G_n(a)$$

(since $a_i \leq 1$ for $i = 1, \ldots, n$), we get

$$\frac{A_n(a)}{A_n(a) + G_n(a)^n} \geq \frac{1}{1 + G_n(a)^{n-1}} \geq \frac{1}{1 + G_n(a)}$$

and then by the inequalities of P. Henrici (see 3.2.40), we get (1).

Reference

Aoyagi, M.: Problem 5333. Amer. Math. Monthly 73, 1022—1023 (1966).

Remark. The inequality (1) is also valid under weaker conditions for a_i, namely when $0 < a_1 \leq \cdots \leq a_n \leq \dfrac{1}{G_{n-1}}$. See D. Borwein's proof of (1) in Amer. Math. Monthly 73, 1023—1024 (1966).

3.2.42 If $a_k > 0$ for $k = 1, \ldots, n$ and $s = \sum_{k=1}^{n} a_k$, then

$$\prod_{k=1}^{n} (1 + a_k) \leq \sum_{k=0}^{n} \frac{s^k}{k!}.$$

Proof. By the arithmetic-geometric mean inequality, we have

$$\prod_{k=1}^{n} (1 + a_k) \leq \left(\sum_{k=1}^{n} \frac{1 + a_k}{n}\right)^n = \left(1 + \frac{s}{n}\right)^n$$

$$= 1 + \sum_{k=1}^{n} 1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{s^k}{k!} \leq \sum_{k=0}^{n} \frac{s^k}{k!}.$$

3.2.43 Let $a_k \geq 1$ for $k = 1, \ldots, n$. Then

$$\prod_{k=1}^{n} (1 + a_k) \geq \frac{2^n}{n+1} \left(1 + \sum_{k=1}^{n} a_k\right). \tag{1}$$

Proof. If $b_k \geq 0$ $(k = 1, \ldots, n)$, then

$$\prod_{k=1}^{n} \left(1 + \frac{b_k}{2}\right) \geq 1 + \frac{1}{2} \sum_{k=1}^{n} b_k \geq 1 + \frac{1}{n+1} \sum_{k=1}^{n} b_k.$$

For $b_k = a_k - 1$, we have (1).

Reference

Brun, V., and G. Arfwedson: Problem 180. Nord. Mat. Tidskr. **8**, 48 (1960).

3.2.44 If n is a nonnegative integer and $x_k > 0$ for $k = 1, \ldots, n+1$, then

$$\sum_{k=1}^{n+1} \frac{1}{1 + x_k} \geq n \Rightarrow \prod_{k=1}^{n+1} \frac{1}{x_k} \geq n^{n+1}.$$

Remark. This problem was proposed by J. Berkes and the solution of C. Bindschedler was published in Elem. Math. **14**, 132 (1959). However, that solution is not simple.

3.2.45 If $x_k \geq 0$ $(k = 1, \ldots, n)$ and $\sum_{k=1}^{n} (1 + x_k)^{-1} \leq 1$, then $\sum_{k=1}^{n} 2^{-x_k} \leq 1$.

Reference

Whittaker, J. V.: Problem 4712. Amer. Math. Monthly **64**, 677–678 (1957).

3.2.46 If $n > 1$ is an integer and $a_k > 0$ for $k = 1, \ldots, n$, then

$$n \left(\sum_{k=1}^{n} \frac{a_k}{s - a_k}\right)^{-1} \leq n - 1 \leq n^{-1} \sum_{k=1}^{n} \frac{s - a_k}{a_k},$$

and

$$\sum_{k=1}^{n} \frac{s}{s - a_k} \geq \frac{n^2}{n-1} \quad \text{with} \quad s = \sum_{k=1}^{n} a_k.$$

References

Trzeciakiewicz, L., and S. Tomaszczyk: Problem 537. Matematyka **11**, 43 (1958), and **12**, 296–297 (1959).
Paasche, I., H. Kummer and C. Blatter: Problem 365. Elem. Math. **15**, 138–139 (1960).

3.2.47 Let $\sum_{i=1}^{n} x_i = na$, with $x_1, \ldots, x_n$ nonnegative. If $k \geq 1$ is an integer, then

$$\sum_{\nu_1, \ldots, \nu_n = 0}^{k} \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{\nu_1! \cdots \nu_n!} \leq n(n^k - 1) \frac{a^{k+1}}{(k+1)!}, \tag{1}$$

where $\sum_{i=1}^{n} \nu_i = k + 1$.

Proof. The multinomial expansion can be written in the form

$$(2)\qquad \sum_{\substack{\nu_1+\cdots+\nu_n=k+1\\ k\geq\nu_i\geq 0}} \frac{(k+1)!}{\nu_1!\cdots\nu_n!}\, x_1^{\nu_1}\cdots x_n^{\nu_n} = \left(\sum_{i=1}^{n} x_i\right)^{k+1} - \sum_{i=1}^{n} x_i^{k+1}.$$

From the relations between means of different orders, we have for $k \geq 1$,

$$(3)\qquad 0 \leq \sum_{i=1}^{n} x_i^{k+1} - n\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{k+1}.$$

Since $\sum_{i=1}^{n} x_i = na$, by adding (2) and (3) we get (1).

Inequality and proof of G. Kalajdžić.

3.2.48 Let $\sum_{k=1}^{n} x_k^2 = 1$, $\sum_{k=1}^{n} a_k x_k = 0$ and $\sum_{k=1}^{n} a_k^2 > 0$, then

$$\left(\sum_{k=1}^{n} b_k x_k\right)^2 \leq \frac{\sum_{i<j} (a_i b_j - a_j b_i)^2}{\sum_{k=1}^{n} a_k^2}.$$

Reference

Ostrowski 2, p. 290.

3.2.49 If $x_i > 0$ $(i = 1, \ldots, n)$, then

$$\binom{n}{2} \sum_{1\leq i<j\leq n} \frac{1}{x_i x_j} \geq 4\left(\sum_{1\leq i<j\leq n} \frac{1}{x_i + x_j}\right)^2.$$

Equality holds if and only if all x_i are equal.

This inequality is due to D. D. Adamović.

3.2.50 If $0 < b \leq a$, then

$$\frac{1}{8}\,\frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\,\frac{(a-b)^2}{b}.$$

3.2.51 If a, b, c are different real numbers, then

$$3\min(a, b, c) < \Sigma a - (\Sigma a^2 - \Sigma ab)^{1/2} < \Sigma a + (\Sigma a^2 - \Sigma ab)^{1/2} < 3\max(a, b, c),$$

where

$$\Sigma a = a + b + c,\quad \Sigma a^2 = a^2 + b^2 + c^2,\quad \Sigma ab = ab + bc + ca.$$

Proof. The function

$$f(x) = (x - a)(x - b)(x - c) = x^3 - (\Sigma a)\,x^2 + (\Sigma ab)\,x - abc$$

vanishes at a, b, c. Its derivative $f'(x)$ vanishes for those x which satisfy

$$3x^2 - 2(\Sigma a)\,x + \Sigma ab = 0.$$

The roots of this equation lie between $\min(a, b, c)$ and $\max(a, b, c)$. This is equivalent to the proposed inequalities.

Remark. This inequality was proposed by D. S. Mitrinović in Nord. Mat. Tidskr. **9**, 138–139 (1961). The published proof by J. Lohne in this journal is more complicated than the one given above.

3.2.52 Let a, b, c, d, e and f be nonnegative real numbers which satisfy

$$a + b \leq e \quad \text{and} \quad c + d \leq f. \tag{1}$$

Then

$$(ac)^{1/2} + (bd)^{1/2} \leq (ef)^{1/2}.$$

Since one may interchange c and d in (1), another valid inequality is

$$(ad)^{1/2} + (bc)^{1/2} \leq (ef)^{1/2}.$$

Proof. Multiplying the inequalities in (1), one obtains

$$ac + bd + (ad + bc) \leq ef.$$

But

$$2(ad \cdot bc)^{1/2} \leq ad + bc,$$

which means that

$$\begin{aligned}\left((ac)^{1/2} + (bd)^{1/2}\right)^2 &= ac + bd + 2(ad \cdot bc)^{1/2}\\ &\leq ac + bd + (ad + bc) \leq ef.\end{aligned}$$

Reference

Diaz, J. B., and F. T. Metcalf: Complementary inequalities III. Math. Ann. **162**, 120–139 (1965). — See, in particular, p. 133.

3.2.53 Let $a_m, \ldots, a_n$ $(m < n)$ be real numbers. Then

$$\left|\sum_{k=m}^{n} k^{-1/3} a_k\right|^3 \leq \left(\sum_{k=m}^{n} |a_k|^{3/2}\right)^2 \sum_{k=m}^{n} k^{-1}.$$

Reference

Ostrowski 2, p. 44.

3.3 Inequalities Involving Polynomials

A lot has been written on bounds of the zeros of a polynomial. Various estimates for polynomials of LEGENDRE, LAGUERRE, HERMITE, ČEBYŠEV and for other polynomials arising in the Theory of Special Functions are also known. From all this, we have only included here some results which seemed especially interesting to us.

In book [1] of G. SZEGÖ, [2] of M. MARDEN and [3] of A. F. TIMAN one can find almost all the more important inequalities related to the topics mentioned above. These books contain a large number of references concerning the topics in question. See also [4].

References

1. SZEGÖ, G.: Orthogonal Polynomials. Amer. Math. Soc. Coll. Publications, vol. **23**. Rev. ed., Providence 1959.
2. MARDEN, M.: Geometry of Polynomials. Amer. Math. Soc. Math. Surveys, vol. **3**. 2nd ed., Providence 1966.
3. TIMAN, A. F.: Theory of Approximation of Functions of a Real Variable. Oxford-London-New York-Paris 1963.
4. DIEUDONNÉ, J.: Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe. Ann. École Norm. Sup. **(3) 48**, 247–258 (1931).

3.3.1 If $ax^2 + bx + c$ is a polynomial with real coefficients and real roots, then

$$a + b + c \leq \frac{9}{4} \max(a, b, c). \tag{1}$$

This inequality is due to L. MOSER and J. R. POUNDER. J. D. DIXON gave the following generalization of (1).

For all real polynomials $P(x) = a_0 + a_1x + \cdots + a_nx^n$ of degree n, with only real zeros, we have

$$a_0 + a_1 + \cdots + a_n \leq \alpha_n \max_k a_k,$$

and

$$\min_k a_k \leq \beta_n \max_k a_k,$$

where

$$\alpha_n = \frac{(n+1)^n}{\binom{n}{s}(n-s)^{n-s}(s+1)^s}, \quad \beta_n = \binom{n}{s}^{-1} \quad \text{with} \quad s = \left[\frac{n}{2}\right].$$

The constants α_n and β_n are the best possible.

References

MOSER, L., and J. R. POUNDER: Problem 53. Canad. Math. Bull. **5**, 70 (1962).
DIXON, J. D.: Polynomials with real roots. Canad. Math. Bull. **5**, 259–263 (1962).

3.3.2 Let x_1, x_2, x_3 be roots of the equation

$$x^3 + px + q = 0 \qquad \text{(where } p \text{ and } q \text{ are real numbers)}$$

and define d and D by

$$d = \min(|x_2 - x_3|, |x_3 - x_1|, |x_1 - x_2|), \quad D = \frac{q^2}{4} + \frac{p^3}{27}.$$

Then the following statements are valid:

1° If $D < 0$, then $2(-3D/|p|)^{1/2} < d \leq 3(-3D/|p|)^{1/2} \leq \sqrt{|p|}$;

2° If $D = 0$, then $d = 0$;

3° If $D > 0$ and $p > 0$, then $p^{1/2} \leq d \leq (p + 12(q^2/4)^{1/3})^{1/2}$;

4° If $D > 0$ and $p < 0$, then

$$\sqrt{3}\,(-q/2)^{1/3} - (-p)^{1/2} < d < 2(p + 3(q^2/4)^{1/3})^{1/2}.$$

Reference

Wrona, W.: On minimal distance between roots of the equation of third degree (Polish). Zeszyty Nauk. Wyż. Szkoł. Ped. Katowice **1966**, No. 5, 9–12.

3.3.3 A necessary and sufficient condition that

$$x^3 + ax^2 + bx + 1 > 0 \quad \text{for} \quad x \geq 0$$

is given by

$$ab + 6(a + b) + 9 + 2(a + b + 3)^{3/2} > 0$$

(a and b real numbers and $a + b + 3 \geq 0$).

Reference

Verblunsky, S.: A theorem on cubic polynomials. Arch. Math. (Basel) **2**, 281–282 (1950).

3.3.4 Let $P(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$ be a real polynomial with $a_0 > 0$, $|a_3| + |a_4| > 0$. Put

$$E = a_0a_4 + 3a_2^2 - 4a_1a_3,$$

$$F = a_0a_3^2 + a_1^2a_4 + a_2^3 - a_0a_2a_4 - 2a_1a_2a_3,$$

$$S = a_0a_4 + 2a_1a_3,$$

$$D = E^3 - 27F^2,$$

$$R = S^2 - 9a_0a_2^2a_4.$$

Then P is nonnegative if and only if $D \geq 0$ and either $R \leq 0$ and $a_2 \geq 0$, or $R \geq 0$ and $-\frac{1}{2}a_0a_4 \leq a_1a_3 < a_0a_4$.

Reference

MAŘÍK, J.: Real polynomials of the 4-th degree. Časopis Pěst. Mat. **90**, 33–42 (1965).

3.3.5 If the roots of the equation

$$a_0x^n - \binom{n}{1} a_1x^{n-1} + \binom{n}{2} a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive and distinct, then

$$a_p a_q > a_r a_s$$

for all p, q, r, s with $p + q = r + s$ and $|p - q| < |r - s|$. In particular,

$$a_p a_{n-p} > a_0 a_n \qquad (p = 1, \ldots, n-1).$$

Reference

OLDS, C. D., and E. P. STARKE: Problem E 1143. Amer. Math. Monthly **62**, 445–446 (1955).

3.3.6 If all the zeros $x_1, \ldots, x_n$ of a real polynomial

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

are positive, then

$$\frac{a_1a_{n-1}}{a_0a_n} \geq n^2.$$

Proof. By the arithmetic-harmonic mean inequality, we have

$$\frac{a_{n-1}}{a_n} \cdot \frac{a_1}{a_0} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \frac{1}{x_j} \geq n^2.$$

Reference

FRANSÉN, A., and J. LOHNE: Problem 196. Nord. Mat. Tidskr. **8**, 188 (1960).

3.3.7 Suppose that a polynomial

$$P(x) = a_0 + a_1x + \cdots + a_nx^n \qquad (a_n \neq 0)$$

has all its zeros in $|x| < 1$. Then

$$\frac{\sum_{k=0}^{n} k\,|a_k|^2}{\sum_{k=0}^{n} |a_k|^2} > \frac{n}{2}.$$

Reference

NEWMAN, D. J.: Problem 5040. Amer. Math. Monthly **69**, 670 (1962).

3.3.8 Let

$$P(x) = a_0 + a_1x + \cdots + a_nx^n, \quad Q(x) = b_0 + b_1x + \cdots + b_mx^m$$
$$(a_n > 0, \quad b_m > 0)$$

be real polynomials all of whose roots are real. If these roots separate each other, then, for $m = n$ and $k = 1, \ldots, n$,

$$a_{k-1}b_k - a_kb_{k-1} > 0,$$

and, for $m = n - 1$ and $k = 1, \ldots, n - 1$,

$$a_{k-1}b_k - a_kb_{k-1} < 0.$$

Reference

CONSTANTINESCU, F.: Relations entre les coefficients de deux polynomes dont les racines se séparent. Časopis Pěst. Mat. **89**, 1–4 (1964).

3.3.9 Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that $|P(z)| \leq M$ for $|z| = 1$. C. VISSER proved that

$$|a_0| + |a_n| \leq M.$$

Q. I. RAHMAN has given the following generalization:

Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and $Q(z) = \sum_{\nu=0}^{m} b_\nu z^\nu$ be polynomials of degrees n and m respectively such that $|P(z)| \leq |Q(z)|$ for $|z| = 1$. If $Q(z)$ has all its zeros in the closed exterior of the unit disc $|z| \leq 1$, then

$$|a_0| + |a_n| \leq |b_m| \quad \text{for} \quad m < n, \tag{1}$$

$$|a_0| + |a_n| \leq |b_0| + |b_m| \quad \text{for} \quad m = n, \tag{2}$$

$$\max(|a_0|, |a_n|) \leq |b_0| - |b_m| \quad \text{for} \quad m > n. \tag{3}$$

If, on the other hand, $Q(z)$ has all its zeros in the closed interior of the unit disc, then the following inequalities

$$|a_0| + |a_n| \leq |b_0| \quad \text{for} \quad m < n,$$

$$\max(|a_0|, |a_n|) \leq |b_m| - |b_0| \quad \text{for} \quad m > n,$$

together with (2) hold.

Reference

RAHMAN, Q. I.: Inequalities concerning polynomials and trigonometric polynomials. J. Math. Anal. Appl. **6**, 303–324 (1963).

3.3.10 If $P(x)$ is a nonzero polynomial with integer coefficients, and if $P(1) = 0$ and $P(2) = 0$, then a coefficient of $P(x)$ is ≤ -2.

Reference

FLATTO, L., N. C. HSU and A. G. KONHEIM: Problem E 1493. Amer. Math. Monthly **69**, 568 (1962).

3.3.11 Let $az^2 + bz + c$ be a polynomial with nonzero complex coefficients. Then the zeros of this polynomial lie in the closed disk

$$|z| \leq \left|\frac{b}{a}\right| + \left|\frac{c}{b}\right|. \tag{1}$$

Proof. Since

$$\begin{aligned} \left|\sqrt{b^2 - 4ac}\right| &= |b| \left|\sqrt{1 - \frac{4ac}{b^2}}\right| \leq |b| \left|\sqrt{1 + \left|\frac{4ac}{b^2}\right|}\right| \\ &\leq |b| \left(1 + \left|\frac{2ac}{b^2}\right|\right) \\ &= |b| + \left|\frac{2ac}{b}\right|, \end{aligned}$$

starting with

$$z = -\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac}$$

one obtains

$$|z| \leq \left|\frac{b}{2a}\right| + \left|\frac{b}{2a}\right| + \left|\frac{c}{b}\right|,$$

i.e., (1).

3.3.12 Let $z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a polynomial with nonzero complex coefficients, and let z_i $(i = 1, \ldots, n)$ be the zeros of this polynomial. Then

$$r_0 \leq \max\left(2|a_1|,\ 2\left|\frac{a_2}{a_1}\right|, \ldots,\ 2\left|\frac{a_{n-1}}{a_{n-2}}\right|,\ \left|\frac{a_n}{a_{n-1}}\right|\right),$$

where

$$r_0 = \max_{1 \leq i \leq n} |z_i|.$$

Reference

BOURBAKI, N.: Topologie générale. 2nd ed., Paris 1955, Chap. 5—8, p. 97, Exerc. 1.

3.3.13 If $x_1, \ldots, x_n$ are roots of the polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

then

$$|x_k| \leq |a_1| + |a_2|^{1/2} + \cdots + |a_n|^{1/n} \qquad (k = 1, \ldots, n).$$

Remark. This result is due to J. L. WALSH [1]. For a generalization, see paper [2] of J. RUDNICKI.

References

1. WALSH, J. L.: An inequality for the roots of an algebraic equation. Ann. of Math. **(2) 25**, 285–286 (1924).
2. RUDNICKI, J.: Remarque sur un théorème de Mr. Walsh. Mathematica (Cluj) **8**, 136–138 (1934).

3.3.14 If $x_1, \dots, x_n$ are roots of the polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

then, for $k = 1, \dots, n$,

$$|x_k| < (1 + |a_1|^2 + \cdots + |a_n|^2)^{1/2} \tag{1}$$

and

$$|x_k| < (1 + |a_1 - 1|^2 + |a_2 - a_1|^2 + \cdots + |a_n - a_{n-1}|^2 + |a_n|^2)^{1/2}. \tag{2}$$

Remark. Inequality (1) is due to R. D. CARMICHAEL and T. E. MASON [1], while (2) is due to K. P. WILLIAMS [2].

References

1. CARMICHAEL, R. D., and T. E. MASON: Note on the roots of algebraic equations. Bull. Amer. Math. Soc. **21**, 14–22 (1914).
2. WILLIAMS, K. P.: Note concerning the roots of an equation. Bull. Amer. Math. Soc. **28**, 394–396 (1922).

3.3.15 If α_k, β_k and λ_k, for $k = 1, \dots, n$, are real numbers with $\beta_k > 0$ and $\lambda_k > 0$ for $k = 1, \dots, n$, then

$$\min_{1 \le i \le n} \left(\frac{\alpha_i}{\beta_i}\right) \le \frac{\sum_{i=1}^{n} \alpha_i \lambda_i}{\sum_{i=1}^{n} \beta_i \lambda_i} \le \max_{1 \le i \le n} \left(\frac{\alpha_i}{\beta_i}\right). \tag{1}$$

Equality holds if and only if the sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are proportional.

This result is a consequence of CAUCHY's inequalities given in 3.2.25.

D. MARKOVIĆ in several papers used (1) for obtaining the bounds for moduli of the zeros of a polynomial. We give a result of his [1].

Consider $P(z) = \sum_{k=0}^{n} a_k z^k$ and $f(z) = \sum_{k=0}^{+\infty} b_k z^k$, with $b_k > 0$ for $k = 0, 1, \dots$ Let r_0 be a positive zero of $Mf(r) = |a_0|$, where $M = \max_{1 \le k \le n} \left(\frac{|a_k|}{b_k}\right)$. Then all the zeros of P lie in $|z| \ge r_0$.

Hint for the proof. For each zero $z = re^{i\theta}$ of P we have

$$\frac{|a_0|}{f(r)} \le \frac{\sum_{k=1}^{n} \frac{|a_k|}{b_k} b_k r^k}{\sum_{k=1}^{n} b_k r^k} \le M.$$

In the particular case when $b_k = t^{-k}$ for $k = 1, 2, \ldots,$ and $g(t) = \max_{1 \le k \le n} (|a_k|\, t^k)$, where t is any positive number, we have that all the zeros of P lie in the domain $|z| \ge \frac{|a_0|\, t}{|a_0| + g(t)}$.

The same result was also given by E. LANDAU [2] in another way.

D. M. SIMEUNOVIĆ [3], assuming that $\lambda_1 > \cdots > \lambda_n > 0$, improved (1) and his inequalities read:

$$\min_{1 \le i \le n} \left(\frac{\alpha_i}{\beta_i}\right) \le \min_{1 \le i \le n} \left(\frac{\sum_{k=1}^{i} \alpha_k}{\sum_{k=1}^{i} \beta_k}\right) \le \frac{\sum_{i=1}^{n} \alpha_i \lambda_i}{\sum_{i=1}^{n} \beta_i \lambda_i} \le \max_{1 \le i \le n} \left(\frac{\sum_{k=1}^{i} \alpha_k}{\sum_{k=1}^{i} \beta_k}\right) \le \max_{1 \le i \le n} \left(\frac{\alpha_i}{\beta_i}\right).$$

Using the above inequalities he showed that all the zeros of P lie in the domain

$$|z| \ge \frac{|a_0|\, t}{|a_0| + h(t)} \quad \text{with} \quad h(t) = \max_{1 \le i \le n} \left(\frac{\sum_{k=1}^{i} |a_k|\, t^k}{i}\right) \le \max_{1 \le k \le n} (|a_k|\, t^k).$$

References

1. MARKOVITCH, D.: Sur la limite inférieure des modules des zéros d'un polynôme. Acad. Serbe Sci. Publ. Inst. Math. **2**, 236–242 (1948).
2. LANDAU, E.: Über eine Aufgabe aus der Funktionentheorie. Tôhoku Math. J. **5**, 97–116 (1914).
3. SIMEUNOVIĆ, D. M.: Sur les limites des modules des zéros des polynômes. Mat. Vesnik **4 (19)**, 293–298 (1967).

3.3.16 Let $a_1, \ldots, a_n$ be real nonnegative numbers such that $\sum_{i=1}^{n} a_i > 0$. Let t be a positive root of the equation

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \tag{1}$$

and let $\lambda_1, \ldots, \lambda_{n-1}$ be arbitrary positive numbers. Then

$$t \le \max\left(\lambda_1, \ldots, \lambda_{n-1}, \left(a_1 + \frac{a_2}{\lambda_1} + \cdots + \frac{a_n}{\lambda_{n-1}^{n-1}}\right)\right). \tag{2}$$

Proof. Let $\lambda = \max(\lambda_1, \ldots, \lambda_{n-1})$. If $\lambda \ge t$, then (2) holds. Let $\lambda < t$. Then $\lambda_1, \ldots, \lambda_{n-1} < t$, and therefore

$$a_1 + \frac{a_2}{\lambda_1} + \cdots + \frac{a_n}{\lambda_{n-1}^{n-1}} \ge a_1 + \frac{a_2}{t} + \cdots + \frac{a_n}{t^{n-1}} = \frac{a_1 t^{n-1} + \cdots + a_n}{t^{n-1}} = t$$

and inequality (2) is true.

The above interesting proof is due to S. B. PREŠIĆ.

3.3.17 Let $f(z) = z^n + p_1 z^{n-1} + \cdots + p_n$ be a polynomial with zeros $z_1, \ldots, z_n$, in an arbitrary order.

E. Landau and W. Specht proved that

$$|z_1 \cdots z_t|^2 \leq 1 + \sum_{\nu=1}^{n} |p_\nu|^2,$$

where $1 \leq t \leq n$.

J. Vicente Gonçalves found the following improvement:

$$|z_1 \cdots z_t|^2 + |z_{t+1} \cdots z_n|^2 \leq 1 + \sum_{\nu=1}^{n} |p_\nu|^2,$$

with $1 \leq t \leq n$. This result is the best possible.

A. M. Ostrowski made a generalization as follows:

Partition of the set of zeros of $f(z)$ into k nonempty sets,

$$\{z_1^{(1)}, \ldots, z_{m_1}^{(1)}\}; \ldots; \{z_1^{(k)}, \ldots, z_{m_k}^{(k)}\},$$

with $m_1 + \cdots + m_k = n$. Then, for any $\lambda \geq 2$,

$$|z_1^{(1)} \cdots z_{m_1}^{(1)}|^\lambda + \cdots + |z_1^{(k)} \cdots z_{m_k}^{(k)}|^\lambda$$
$$\leq \left(1 + \sum_{\nu=1}^{n} |p_\nu|^2\right)^{\lambda/2} + k - 2,$$

where equality occurs if and only if $\lambda = k = 2$ or, for $\lambda = 2$ if $k - 2$ of the left-hand terms are equal to 1.

References

Landau, E.: Sur quelques théorèmes de M. Petrović relatifs aux zéros des fonctions analytiques. Bull. Soc. Math. France **33**, 251–261 (1905).

Landau, E.: Über eine Aufgabe aus der Funktionentheorie. Tôhoku Math. J. **5**, 97–116 (1914).

Specht, W.: Abschätzungen der Wurzeln algebraischer Gleichungen. Math. Z. **52**, 310–321 (1949).

Vicente Gonçalves, J.: L'inégalité de W. Specht. Univ. Lisboa Revista Fac. Ci. **(2) A 1**, 167–171 (1950).

Ostrowski, A. M.: On an inequality of J. Vicente Gonçalves. Univ. Lisboa Revista Fac. Ci. **(2) A 8**, 115–119 (1960).

3.3.18 Let P be a real polynomial defined by

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Then

$$\max_{-1 \leq x \leq 1} |P(x)|^2 - \min_{-1 \leq x \leq 1} |P(x)|^2 \geq \frac{|a_n|^2}{2^{n-2}}.$$

This inequality contains Čebyšev's inequality

$$\max_{-1 \leq x \leq 1} |P(x)| \geq \frac{|a_n|}{2^{n-1}}.$$

Reference

van der Corput, J. G., and C. Visser: Inequalities concerning polynomials and trigonometric polynomials. Indagationes Math. **8**, 238—247 (1946).

3.3.19 Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial of degree n whose roots are $x_1, \ldots, x_n$. If none of $\operatorname{Im} x_k$ is equal to zero, then for all real x,

$$\sum_{k=1}^{n} \left(\frac{1}{\operatorname{Im} x_k}\right)^2 \geq -8 \operatorname{Re} \frac{P(x)\, P''(x) - P'(x)^2}{P(x)^2}.$$

If none of $\operatorname{Re} x_k$ is equal to zero, then for all real x,

$$\sum_{k=1}^{n} \left(\frac{1}{\operatorname{Re} x_k}\right)^2 \geq 8 \operatorname{Re} \frac{P(ix)\, P''(ix) - P'(ix)^2}{P(ix)^2}.$$

The above inequalities are equivalent respectively to

$$\sum_{k=1}^{n} \left(\frac{1}{\operatorname{Im} x_k}\right)^2 \geq 8 \operatorname{Re} \frac{a_1^2 - 2a_0 a_2}{a_0^2}, \quad \sum_{k=1}^{n} \left(\frac{1}{\operatorname{Re} x_k}\right)^2 \geq 8 \operatorname{Re} \frac{2a_0 a_2 - a_1^2}{a_0^2}.$$

Reference

Ostrowski, A.: Note sur les parties réelles et imaginaires des racines des polynomes. J. Math. Pures Appl. **(9) 44**, 327—329 (1965).

3.3.20 The complex polynomial $P(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$ has r zeros whose moduli are greater than 1 and $n - r$ zeros whose moduli are smaller than 1 if

$$|a_r| > |a_0| + \cdots + |a_{r-1}| + |a_{r+1}| + \cdots + |a_n|.$$

Remark. This result is due to D. E. Mayer [1]. A simple proof was given by M. Tajima in [2].

References

1. Mayer, D. E.: Sur les équations algébriques. Nouv. Ann. Math. **(3) 10**, 111—124 (1891).
2. Tajima, M.: On the roots of an algebraic equation. Tôhoku Math. J. **19**, 173—174 (1921).

3.3.21 Let $P(x) = (x - a_1) \cdots (x - a_n)$, where $a_1, \ldots, a_n$ are real numbers with $a_k < a_{k+1}$ for $k = 1, \ldots, n - 1$. Let

$$P'(x) = (x - c_1) \cdots (x - c_{n-1})$$

and let $a_k < c_k < a_{k+1}$ for $k = 1, \ldots, n - 1$. Then, for $k = 1, \ldots, n - 1$,

$$a_k + \frac{a_{k+1} - a_k}{n - k + 1} \leq c_k \leq a_{k+1} - \frac{a_{k+1} - a_k}{k + 1}.$$

Reference

Peyser, G.: On the roots of the derivative of a polynomial with real roots. Amer. Math. Monthly **74**, 1102—1104 (1967).

3.3.22 If a polynomial P of degree n has no zeros in the circle $|z| \leq a + b$, where $a \geq 1$ and $b > 0$, then for $-a \leq x \leq a$,

$$|P(x)| \geq \left(\frac{b}{a+b-1}\right)^n \left| P\left(\frac{x}{a}\right) \right|.$$

This inequality, together with a number of inequalities which are analogous to it, has been proved by D. I. MAMEDHANOV.

Reference

MAMEDHANOV, D. I.: Some inequalities for algebraic polynomials and rational functions (Russian). Izv. Akad. Nauk Azerbaidžan. SSR Ser. Fiz.-Mat. Tehn. Nauk 1962, No. 5, 9–15.

3.3.23 Let $P(z)$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$ and let $z'_1, \ldots, z'_{n-1}$ be zeros of its derivative $P'(z)$. Then, for $p \geq 1$,

$$\frac{1}{n} \sum_{k=1}^{n} |\operatorname{Im} z_k|^p \geq \frac{1}{n-1} \sum_{k=1}^{n-1} |\operatorname{Im} z'_k|^p, \quad \frac{1}{n} \sum_{k=1}^{n} |z_k|^p \geq \frac{1}{n-1} \sum_{k=1}^{n-1} |z'_k|^p.$$

Reference

DE BRUIJN, N. G., and T. A. SPRINGER: On the zeros of a polynomial and of its derivative II. Indagationes Math. 9, 264–270 (1947).

3.3.24 For all real x and for any even natural number n,

$$P(x) = x^n - nx + n - 1 \geq 0,$$

equality holding if and only if $x = 1$.

Proof. According to DESCARTES' rule of signs, the polynomial P cannot have more than two positive zeros and can have no negative zeros. This polynomial has a double zero at $x = 1$, and those are, therefore, its only real zeros. This implies the above inequality.

Remark 1. This elementary inequality, indicated in HARDY-LITTLEWOOD-POLYA's book on p. 61 (Theorem 60), has been used in a number of cases as a starting point in the process of finding other inequalities. (See, for example, BENSON's method, in 2.19).

Remark 2. $P(x) \geq 0$ holds for all $x \geq 0$ when n is any real number >1 (with equality if and only if $x = 1$), and for all $x \geq -1$ when n is an odd integer >1. These extensions are sometimes useful.

3.3.25 Let $P(x)$ be a polynomial of degree $n \geq 2$ all of whose roots are real and which satisfies the following conditions:

$$P(-1) = P(+1) = 0, \quad P(x) \neq 0 \quad \text{for} \quad -1 < x < +1$$

$$\text{and} \max_{-1<x<+1} P(x) = 1.$$

Then, if $P(a) = P(b) = d \leq 1$ $(-1 < a < b < +1)$, we have

$$b - a \leq 2(1-d)^{1/2},$$

with equality if and only if $P(x) = 1 - x^2$.

Reference

ERDÖS, P.: Note on some elementary properties of polynomials. Bull. Amer. Math. Soc. **46**, 954–958 (1940).

3.3.26 Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial, where a_0 is real, and $|\operatorname{Re} P(z)| \leq 1$ for $|z| \leq 1$.

Then, for $|z| \leq 1$,

$$|\operatorname{Im} P(z)| \leq \frac{2}{n+1} \sum_{k=1}^{[(1/2)(n+1)]} \cot \frac{(2k-1)\pi}{2n+2}.$$

References

SZEGÖ, G.: On conjugate trigonometric polynomials. Amer. J. Math. **65**, 532–536 (1943).

MULHOLLAND, H. P.: On two extremum problems for polynomials on the unit circle. J. London Math. Soc. **31**, 191–199 (1956).

3.3.27 Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real polynomial of degree $n \geq 0$. Then

$$\min_{0 \leq k \leq n} b_k \leq P(x) \leq \max_{0 \leq k \leq n} b_k \qquad (0 \leq x \leq 1),$$

where

$$b_k = \sum_{r=0}^{k} a_r \binom{k}{r} \Big/ \binom{n}{r} \qquad (k = 0, 1, \ldots, n).$$

Reference

CARGO, G. T., and O. SHISHA: The Bernstein form of a polynomial. J. Res. Nat. Bur. Standards Sect. B, **70 B**, 79–81 (1966).

3.3.28 Let F denote the set of all quadratic polynomials $P(x)$ satisfying

$$P(x) \geq 0\,(-1 \leq x \leq +1) \quad \text{and} \quad \int_{-1}^{+1} P(x)\,dx = 1.$$

Then

$$\min_{-1 \leq \xi \leq 1} \left(\max_{F} P(\xi) \right) = \frac{2}{3}.$$

Reference

SCHOENBERG, I. J.: Problem 28. Wisk. Opgaven **21**, 28–30 (1960).

3.3.29 If all zeros of a real polynomial P of degree n are real, then

$$(n-1)\,P'(x)^2 - nP(x)\,P''(x) \geq 0,$$

where P' and P'' denote the derivatives of P.

Reference

LOVE, J. B.: Problem E 1532. Amer. Math. Monthly **69**, 668 (1962).

3.3.30 If the moduli of each of the roots of a polynomial with complex coefficients

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

do not exceed a positive number M, then

$$|P^{(k)}(z)| \leq k! \binom{n}{k} |a_0| (|z| + M)^{n-k},$$

where $k = 0, 1, \ldots, n$; $P^{(0)}(z) = P(z)$.

Reference

Colucci, A.: Generale maggiorazione dei polinomi e delle derivate e una sua conseguenza. Boll. Un. Mat. Ital. (3) 8, 258–260 (1953).

3.3.31 Let P denote a polynomial in x, and let deg P be the degree of P.

(S. Bernstein.) If deg $P = n$, then

$$\frac{|P'(x)|}{\max |P(x)|} \leq \frac{n}{\sqrt{(b-x)(x-a)}} \qquad (a \leq x \leq b).$$

If deg $P = 2n + 1$ and P is increasing in $a \leq x \leq b$, then

$$\frac{|P(x)|}{\max |P'(x)|} \geq \frac{1}{2} \frac{b-a}{(n+1)^2} \qquad (a \leq x \leq b).$$

(A. Markoff.) If deg $P = n$, then

$$\frac{|P'(x)|}{\max |P(x)|} \leq \frac{2n^2}{b-a} \qquad (a \leq x \leq b).$$

(I. Schur.) If deg $P = n \geq 2$ and if $P(a) = P(b) = 0$, then

$$\frac{|P'(x)|}{\max |P(x)|} \leq \frac{2n \cot \dfrac{\pi}{2n}}{b-a} \qquad (a \leq x \leq b).$$

(P. Erdös.) If deg $P = n$ and P has only real zeros, none in (a, b), then

$$\frac{|P'(x)|}{\max |P(x)|} < \frac{en}{b-a} \qquad (a \leq x \leq b).$$

This is the best possible result.

(S. Bernstein.) Let P be a complex polynomial in complex z with $|P(z)| \leq 1$ for $|z| \leq 1$. Then

$$|P'(z)| \leq n \quad \text{for} \quad |z| \leq 1.$$

(P. D. Lax.) If $|P(z)| \leq 1$ for $|z| \leq 1$ and if P has no zeros inside $|z| = 1$, then

$$|P'(z)| \leq \frac{n}{2} \quad \text{for} \quad |z| \leq 1.$$

The following results are due to A. B. Soble.

1° If the coefficients of P are real and nonnegative, then

$$\frac{P'(x)}{P(x)} \leq \frac{n}{x} \qquad (x > 0).$$

(The case $P(x) = x^n$ shows that this is the best possible result). If, moreover, the constant term is zero, then

$$\frac{P'(x)}{P(x)} \geq \frac{1}{x} \qquad (x > 0).$$

(The case $P(x) = x$ shows that this is the best possible result.)

2° Let

$$P(x) = \sum_{k=0}^{n} c_k x^{n-k} \qquad (c_k > 0).$$

If $0 < c \leq c_k/c_{k-1}$ $(k = 1, \ldots, n)$, where c is a fixed number, then

$$\frac{P'(x)}{P(x)} \leq \frac{n}{2x} \qquad (0 < x \leq c).$$

The constant $\frac{n}{2}$ is the best possible.

3° Let $P(x) = x^n + \sum_{k=1}^{n} c_k x^{n-k}$. If $|c_k| \leq c - 1$, $c > 1$ $(k = 1, \ldots, n)$, then

$$\left|\frac{P'(x)}{P(x)}\right| \leq n \frac{(x - c/2)^2}{x(x - c)^2} \qquad (x > c).$$

4° Let $P(x) = \sum_{k=0}^{n} c_k x^{n-k}$ $(c_k > 0)$. If $c_k \geq c_{k-1}$ $(k = 1, \ldots, n)$, then

$$\frac{P'(x)}{P(x)} \leq \frac{n+1}{sx} \qquad (0 < x \leq \exp(-s/e) \text{ and } s \geq e).$$

References

Schaeffer, A. C.: Inequalities of A. Markoff and S. Bernstein for polynomials and related functions. Bull. Amer. Math. Soc. **47**, 565–579 (1941).

Soble, A. B.: Majorants of polynomial derivatives. Amer. Math. Monthly **64**, 639–643 (1957).

3.3.32 Let P be a complex polynomial of degree $\leq n$. If for $-1 \leq x \leq +1$, $|P(x)| \leq 1$, then, for $-1 \leq x \leq +1$ and $k = 1, \ldots, n$,

$$|P^{(k)}(x)|^2 \leq M_k(x), \tag{1}$$

where

$$M_k(x) = \left(\frac{d^k}{dx^k} \cos nt\right)^2 + \left(\frac{d^k}{dx^k} \sin nt\right)^2 \qquad \text{with } x = \cos t.$$

Remark. Putting $k = 1$ in (1), we get Bernstein's inequality (see 3.3.31). Inequality (1) also contains the following inequality of Markoff:

$$|P^{(k)}(x)| \leq \frac{n^2(n^2 - 1) \cdots (n^2 - (k-1)^2)}{(2k-1)!!} \qquad (-1 \leq x \leq +1).$$

Reference

SCHAEFFER, A. C., and R. J. DUFFIN: On some inequalities of S. Bernstein and W. Markoff for derivatives of polynomials. Bull. Amer. Math. Soc. **44**, 289–297 (1938).

3.3.33 Let $\left|\sum_{k=0}^{n} a_k x^k\right| \leq 1$ for $x \in [-1, +1]$. Then

$$\left|\sum_{k=0}^{n} \log(k+1)\, a_k x^k\right| \leq C \log n,$$

where C is a constant.

Reference

SZEGÖ, G.: On some problems of approximations. Magyar Tud. Akad. Mat. Kutato. Int. Közl. 9, 3–9 (1964).

3.3.34 Consider $\sum_{r=0}^{n} a_r x^r = a_n \prod_{r=1}^{n} (x - x_r)$ with $a_k \neq 0$. Then

$$\sum_{r=0}^{n} |a_r|\, |x|^r \leq |a_n| \prod_{r=1}^{n} (|x| + |x_r|).$$

Reference

OSTROWSKI 3, p. 32.

3.3.35 Let P be a real polynomial of degree $n \geq 2$ having only real zeros, and let P be positive on $(-1, +1)$. Then

$$\int_{-1}^{+1} P(x)\, dx \geq \frac{2}{n+1} \max_{-1 \leq x \leq +1} P(x).$$

If, in addition, $P(-1) = P(+1) = 0$, then

$$\int_{-1}^{+1} P(x)\, dx \geq \frac{2}{n+1}\left(1 + \frac{1}{n-1}\right)^{n-1} \max_{-1 \leq x \leq +1} P(x).$$

These inequalities are still valid when the condition of having only real zeros is replaced by the condition of having no zeros in the open disk $|z| < 1$.

References

SENDOV, B.: An integral inequality for algebraic polynomials with only real zeros (Bulgarian). Ann. Univ. Sofia, Fac. Sci. Phys. Math. Livre 1, Math. 53, 19–32 (1958/59).

DIMOVSKI, I., and V. ČAKALOV: Integral inequalities for real polynomials (Bulgarian). Ann. Univ. Sofia, Fac. Math. 59, 151–158 (1964/65).

3.3.36 Let P be a real polynomial whose derivative P' has only real zeros. Let $P(x) > 0$ for $x \in (-1, +1)$ and let P' have only one zero in $(-1, +1)$.

1° If -1 and $+1$ are two simple zeros of P, then

$$\int_{-1}^{+1} P(x)\,dx \geq \frac{4}{3}\,\frac{P'(1)\,P'(-1)}{P'(1)-P'(-1)}.$$

2° If -1 and $+1$ are two zeros of P, then

$$\int_{-1}^{+1} P(x)\,dx \leq \frac{4}{3}\,P(x_0),$$

where x_0 is the zero of P' which belongs to $(-1, +1)$.

In both cases equality holds if and only if

$$P(x) = c(1-x^2).$$

This result is due to P. Erdös and T. Grünwald.

It was later generalized by H. Kuhn.

References

1. Erdös, P., and T. Grünwald: On polynomials with only real roots. Ann. of Math. **(2) 40**, 537–548 (1939).
2. Erdös, P.: Note on some elementary properties of polynomials. Bull. Amer. Math. Soc. **46**, 954–958 (1940).
3. Iyengar, K. S. K.: A property of integral functions with real roots and of order less than two. Proc. Indian Acad. Sci. **A 12**, 223–229 (1940).
4. Kuhn, H.: Über Polynome mit lauter reellen Nullstellen der Ableitung. Dissertation, Karlsruhe 1966, 40 pp.

3.3.37 Let r denote the number of real zeros (multiplicity of zeros is taken into account) of a polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad (a_0 a_n \neq 0)$$

and let

$$N = \frac{1}{\sqrt{|a_0 a_n|}}\,(|a_0| + |a_1| + \cdots + |a_n|),$$

$$M = \frac{1}{\sqrt{|a_0 a_n|}}\,\max_{|z|=1} |P(z)|.$$

Then, for every integer $n \geq 1$,

$$r^2 - 2r < 4n \log N,$$

$$r(r+1) < 4(n+1)\log N,$$

$$r(r+1) + (p-q)^2 < 4(n+1)\log M,$$

where p and q denote the numbers of positive and negative zeros, respectively, of $P(z)$.

The constant 4 in all above inequalities is the best possible.

References

SCHMIDT, E.: Über algebraische Gleichungen von Pólya-Bloch-Typus. Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1932, pp. 321—328.

SCHUR, I.: Untersuchungen über algebraische Gleichungen. I: Bemerkungen zu einem Satz von E. Schmidt. Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1933, pp. 403—428.

SZEGÖ, G.: Bemerkungen zu einem Satz von E. Schmidt über algebraische Gleichungen. Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1934, pp. 86—98.

3.3.38 The FIBONACCI polynomial F_n, defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad F_1(x) = 1, \quad F_2(x) = x,$$

satisfies

$$F_n(x)^2 \leq (x^2+1)^2 (x^2+2)^{n-3}$$

for $n = 3, 4, \ldots$

Reference

SWAMY, M. N. S.: Problem E 1846. Amer. Math. Monthly 73, 81 (1966).

3.3.39 Let $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be an arbitrary polynomial with complex coefficients. Define

$$H(P) = \max(|a_0|, |a_1|, \ldots, |a_n|),$$

$$L(P) = |a_0| + |a_1| + \cdots + |a_n|,$$

and

$$M(P) = 0 \quad \text{if} \quad P(x) \equiv 0,$$

$$M(P) = \exp\left(\int_0^1 \log|P(e^{2\pi i t})|\,dt\right) \text{ otherwise.}$$

Then, for any two polynomials P and Q,

$$L(PQ) \leq L(P)\,L(Q),$$

$$L(P \mp Q) \leq L(P) + L(Q).$$

The height H, the length L and the measure M of P are connected by the inequalities

$$\binom{n}{[n/2]}^{-1} H(P) \leq M(P) \leq H(P)\sqrt{n+1},$$

$$2^{-n} L(P) \leq M(P) \leq L(P).$$

If both P and Q are at most of degree n, then

$$M(P \mp Q) \leq L(P \mp Q) \leq L(P) + L(Q) \leq 2^n (M(P) + M(Q)).$$

R. L. DUNCAN proved that 2^n in the inequality of K. MAHLER

$$M(P+Q) \le 2^n\bigl(M(P)+M(Q)\bigr)$$

can be replaced by $\binom{2n}{n}^{1/2}$.

Let $L^*(P) = (|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2)^{1/2}$. Then

$$\binom{2n}{n}^{-1/2} L^*(P) \le M(P) \le L(P),$$

$$L^*(PQ) \ge \left(\binom{2m}{m}\binom{2n}{n}\right)^{-1/2} L^*(P)\,L^*(Q),$$

where P and Q are polynomials of degree n and m respectively.

Remark. These inequalities have applications in the theory of transcendental numbers.

References

MAHLER, K.: On two extremum properties of polynomials. Illinois J. Math. **7**, 681–701 (1963).

DUNCAN, R. L.: Some inequalities for polynomials. Amer. Math. Monthly **73**, 58–59 (1966).

3.3.40 Let $x_1, \ldots, x_n$ $(x_1 \le \cdots \le x_n)$ be the zeros of a real polynomial $P(x)$ of degree n and let $y_1, \ldots, y_{n-1}$ $(y_1 \le \cdots \le y_{n-1})$ be the zeros of the derivative $P'(x)$. Put

$$(n-1)\,x_k' = (x_1 + \cdots + x_n) - x_k,$$

and

$$(n-2)\,y_k' = (y_1 + \cdots + y_{n-1}) - y_k.$$

Furthermore, let $x \mapsto f(x)$ be a nonconcave function.

Then, the following inequalities hold:

$$\frac{1}{j}\sum_{k=1}^{j} x_k \ge \frac{1}{j-1}\sum_{k=1}^{j-1} y_k,$$

$$\frac{1}{j}\sum_{k=n-j+1}^{n} x_k \le \frac{1}{j-1}\sum_{k=n-j+1}^{n-1} y_k,$$

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \ge \frac{1}{n-1}\sum_{k=1}^{n-1} f(y_k),$$

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k') \le \frac{1}{n-1}\sum_{k=1}^{n-1} f(y_k').$$

If $x_1, \ldots, x_n$ are real nonnegative numbers and

$$A = \frac{x_1 + \cdots + x_n}{n}, \quad G = \sqrt[n]{x_1 \cdots x_n},$$

then
$$G \leq \sqrt[n]{x_n' \cdots x_n'} \leq A,$$
$$G^2 \leq \binom{n}{2}^{-1} \sum x_i' x_k' \leq A^2,$$
where $i = 1, \ldots, n-1$ and $k = i+1, \ldots, n$.

References

Popoviciu, T.: Sur certaines inégalités entre les zéros, supposés tous réels, d'un polynome et ceux de sa dérivée. Ann. Sci. Univ. Jassy Sect. I, **30**, 191–218 (1944/47).

See also:

Bray, H.: On the zeros of a polynomial and of its derivative. Amer. J. Math. **53**, 864–872 (1931).

Toda, K.: On certain functional inequalities. J. Sci. Hiroshima Univ. Ser. A-I Math. **4**, 27–40 (1934).

3.3.41 Let c_i denote the real zeros of the following real polynomial:
$$P(x) = a_n x^n + a_1 x^{n-1} + \cdots + a_0 \qquad (a_n > 0).$$
Let α_k denote the real parts of its imaginary zeros, $a = \max(c_i, \alpha_k)$, $b = \min(c_i, \alpha_k)$, and let $P^{(m)}(x)$ be the m-th derivative of $P(x)$. Then, unless identically zero, the derivatives of $P(x)$ satisfy the inequalities
$$P^{(m)}(x) < 0 \quad \text{for} \quad x > a;$$
$$(-1)^r P^{(2m)}(x) > 0 \text{ and } (-1)^r P^{(2m+1)}(x) < 0 \text{ for } x < b,$$
where r is the number of real zeros of $P(x)$.

Reference

Marković, D.: Sur les zéros réels des dérivées de quelques fonctions (Serbian). Bull. Soc. Math. Phys. Serbie **4**, No. 3–4, 1–5 (1952).

3.3.42 Let P be a real polynomial of total degree n in k real variables. Then
$$\max_{\substack{|x_r| \leq 1 \\ r=1,\ldots,k}} |P(x_1, \ldots, x_k)| \geq \frac{1}{2^{n-1}} \max_{\substack{|x_r| = 1 \\ r=1,\ldots,k}} |P^*(x_1, \ldots, x_k)|,$$
where P^* denotes the sum of the terms in P whose degree is n.

Reference

Visser, C.: A generalization of Tchebychef's inequality to polynomials in more than one variable. Indagationes Math. **8**, 310–311 (1946).

3.3.43 If P and Q are two polynomials in z of degrees m and n respectively, and if E denotes a bounded continuum, then
$$\max_E |P(z)\, Q(z)| \geq C(m, n) \max_E |P(z)| \max_E |Q(z)|,$$
where the constant $C(m, n)$ depends only on m and n.

This result, due to G. AUMANN [1], was improved by H. KNESER [2] who showed that $\prod_{k=1}^{m} \tan^2 \frac{2k-1}{4(m+n)} \pi$ is the exact value of $C(m, n)$, and determined when it is attained.

The above result has the following geometric interpretation: Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ be fixed points in the z-plane, and M a point of E. Then

$$\max_E \left(\prod_{i=1}^{m} MA_i \prod_{i=1}^{n} MB_i \right) \geq C(m, n) \max_E \prod_{i=1}^{m} MA_i \max_E \prod_{i=1}^{n} MB_i.$$

M. BIERNACKI [3] later proved a number of analogous inequalities replacing the products by the sums or other symmetric functions, or replacing fixed points in the plane by other geometric elements and also considering the problem in space.

References

1. AUMANN, G.: Satz über das Verhalten von Polynomen auf Kontinuen. Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1933, pp. 924–931.
2. KNESER, H.: Das Maximum des Produkts zweier Polynome. Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1934, pp. 426–431.
3. BIERNACKI, M.: Sur quelques propriétés des fonctions de distances. J. Math. Pures Appl. **(9) 31**, 305–318 (1952).

3.3.44 Let a, b and c be three points in the complex plane, and let $A(a, b, c)$ denote the area of the triangle determined by those points.

Let P be an arbitrary polynomial of degree n with zeros $z_1, \ldots, z_n$ and let $\zeta_1, \ldots, \zeta_{n-1}$ be the zeros of the derivative P'. Then the following inequality holds

$$\frac{1}{(n-1)^3} \sum A(\zeta_p, \zeta_r, \zeta_s) \leq \frac{1}{n^3} \sum A(z_i, z_j, z_k),$$

where $1 \leq i < j < k \leq n$ and $1 \leq p < r < s \leq n-1$ are the ranges of summation.

Reference

KRAMER, H.: Über einige Ungleichungen zwischen den Nullstellen eines Polynoms und seiner ersten Ableitung. Mathematica (Cluj) **10 (33)**, 89–93 (1968).

3.4 Inequalities Involving Trigonometric Functions

3.4.1 If $0 \leq a < b \leq \pi/2$, then

$$\frac{a}{b} \leq \frac{\sin a}{\sin b} \leq \frac{\pi}{2} \frac{a}{b}.$$

Reference

GARNIR, H. G.: Fonctions de variables réelles, vol. 1. Louvain-Paris 1963, p. 267.

3.4.2 If $n \geq 3$ and $\frac{\pi}{n} \leq x \leq \pi - \frac{\pi}{n}$, then

$$\frac{\sin x}{3} + \frac{\sin nx}{2n} > 0.$$

3.4.3 If $x \geq 1$, then

$$\sin\frac{1}{x-1} - 2\sin\frac{1}{x} + \sin\frac{1}{x+1} > 0. \tag{1}$$

Proof. If $y = \sin\frac{1}{x}$, then

$$y'' = \frac{1}{x^4}\left(2x - \tan\frac{1}{x}\right)\cos\frac{1}{x} > 0 \qquad (x \geq 1).$$

So, y is a convex function of x on $[1, +\infty)$ and (1) follows from JENSEN's inequality.

3.4.4 If $0 < x_k < \pi$ for $k = 1, 2, \ldots$, and if $n > 1$ is an integer, then

$$\left|\sin \sum_{k=1}^{n} x_k\right| < \sum_{k=1}^{n} \sin x_k.$$

3.4.5 If x is real with $0 < |x| < \pi$, and $|r - s| < 1/2$, where r and s are real, then

$$\left|\frac{\sin rx}{2\sin(x/2)} - \frac{\sin sx}{x}\right| < 1.$$

Reference

MAKAI, E.: On the summability of the Fourier series of L^2 integrable functions IV. Acta Math. Acad. Sci. Hung. **20**, 383–391 (1969).

3.4.6 The largest a and smallest b for which

$$\cos bx \leq \frac{\sin x}{x} \leq \cos ax$$

on $(0, \pi/2)$ are

$$a = \frac{2}{\pi}\arccos\frac{2}{\pi}, \quad b = \frac{1}{\sqrt{3}}. \tag{1}$$

The following inequalities also hold:

$$1 \geq \cos\frac{x}{2} \geq \cos ax \geq \frac{\sin x}{x} \geq \cos\frac{x}{\sqrt{3}} \geq \sqrt[3]{\cos x} \geq \frac{\cos x}{1 - x^2/3} \geq \cos x.$$

Reference vdl. S. 270

IYENGAR, K. S. K., B. S. MADHAVA RAO and T. S. NANJUNDIAH: Some trigonometrical inequalities. Half-Yearly J. Mysore Univ. **B** (N.S.) **6**, 1–12 (1945).

3.4.7 If $0 < \beta - \alpha < \pi$, then

$$\max_{\alpha \leq x \leq \beta} |1 + a\cos x + b\sin x| \geq \tan^2\frac{\alpha - \beta}{4}.$$

Reference

GONČAROV, V. L.: Theory of Interpolation and Approximation of Functions (Russian). 2nd ed., Moscow 1954, pp. 233–234.

3.4.8 Let $2 < a \leq 3$. Then

$$\cos\frac{\pi}{a} \leq \cos^2\frac{\pi}{a+1}.$$

For $a \geq 3$, the inequality is reversed.

Reference

LAUFFER, R.: Problem 144. Wisk. Opgaven **19**, 294–295 (1950/54).

3.4.9 If $0 < r < 1/2$, then

$$\cos \pi r > 1 - 2r.$$

If $1/2 < r < 1$, then

$$\cos \pi r < 1 - 2r.$$

Reference

KOBER, H.: Approximation by integral functions in the complex domain. Trans. Amer. Math. Soc. **56**, 22 (1944).

3.4.10 If $a_1, \ldots, a_n$ are real numbers, then

$$\sum_{i<j}^{n} \cos(a_i - a_j) \geq -\frac{n}{2}.$$

3.4.11 If $0 < a < \pi/2$ and if t is an arbitrary real number, then

$$2(\cos a - \sin a)^2 \leq [\cos(a+t) + \sin(a+t)]^2 + [\cos(a-t) + \sin(a-t)]^2 \leq 2(\cos a + \sin a)^2.$$

3.4.12 If $0 \leq a, b \leq \pi$, then

$$|\cos a - \cos b| \geq |a - b| \sqrt{\sin a \sin b}.$$

Reference

VAN DER CORPUT, J. G.: Problem 96. Wisk. Opgaven **16**, 243–244 (1935).

3.4.13 If k is a natural number and if a and b are real numbers such that $\cos a \neq \cos b$, then

$$\left|\frac{\cos ka - \cos kb}{\cos a - \cos b}\right| \leq k^2,$$

$$\left|\frac{\cos kb \cos a - \cos ka \cos b}{\cos b - \cos a}\right| \leq k^2 - 1.$$

Equality holds only in the limit as a and b approach zero.

Reference

GOODMAN, A. W.: Problem E 723. Amer. Math. Monthly **53**, 271 (1946).

3.4.14 Let $0 \leq r < 1$. For all real a and b the following inequality holds:

$$(1) \qquad \frac{(1 - r^2) \cos a + 2r \sin a \sin b}{1 - 2r \cos b + r^2} \geq \frac{(1 + r^2) \cos a - 2r}{1 - r^2}.$$

Proof. Inequality (1) is equivalent to

$$(2) \qquad [2r - (1 + r^2) \cos a] \cos b - [(1 - r^2) \sin a] \sin b \leq 1 - 2r \cos a + r^2.$$

For fixed r and a, the maximum value of the left-hand side of (2) is the positive square root of the expression

$$[2r - (1 + r^2) \cos a]^2 + [(1 - r^2) \sin a]^2 = (1 - 2r \cos a + r^2)^2.$$

Thus (2) and (1) are verified.

For given r and a there exists b for which equality occurs in (2) and (1).

Reference

Robertson, M. S.: Radii of star-likeness and close-to-convexity. Proc. Amer. Math. Soc. **16**, 847–852 (1965).

3.4.15 If $0 < p \leq 1/2$ and $0 \leq x \leq \pi/2$, then

$$\frac{(p + 1) \sin x}{1 + p \cos x} \leq x \leq \frac{\pi}{2} \frac{\sin x}{1 + p \cos x}.$$

Remark. The above inequalities represent a partial answer to the problem E 1277 proposed by A. Oppenheim and solved by W. B. Carver in Amer. Math. Monthly **65**, 206–209 (1958).

3.4.16 If $0 \leq a \leq 1$ and $0 < x \leq \pi/3$, then

$$\sin^2 x \leq 1 - 2a \cos x + a^2 \leq 1,$$

$$\frac{3}{4} \leq \frac{1 - 2a \cos x + a^2}{2(1 - \cos x)} \leq \frac{1}{\sin^2 x}.$$

3.4.17 If $0 < a < \pi/2$, then

$$\sum_{k=1}^{n} \left(\frac{1}{\cos \frac{a}{k}} + \frac{1}{\sin \frac{a}{k}} \right) > \sum_{k=1}^{n} \frac{1}{\sin \frac{a}{k} \cos \frac{a}{k}}.$$

3.4.18 If $a \leq 3$, and $0 < x < \pi/2$, then

$$(1) \qquad \cos x < \left(\frac{\sin x}{x}\right)^a.$$

If $a > 3$, then there exists $x_1 \in (0, \pi/2)$, depending on a, such that

$$\text{(2)}\qquad \begin{aligned} &\cos x > \left(\frac{\sin x}{x}\right)^a &&(0 < x < x_1),\\ &\cos x_1 = \left(\frac{\sin x_1}{x_1}\right)^a, &&\\ &\cos x < \left(\frac{\sin x}{x}\right)^a &&(x_1 < x < \pi/2). \end{aligned}$$

Proof. It is sufficient to prove (1) for $a = 3$. Denoting

$$\text{(3)}\qquad f(x) = x - \sin x\,(\cos x)^{-\frac{1}{a}},$$

we obtain

$$\text{(4)}\qquad f'(x) = 1 - (\cos x)^{\frac{a-1}{a}} - \frac{1}{a}\sin^2 x\,(\cos x)^{-\frac{a+1}{a}},$$

$$\text{(5)}\qquad f''(x) = \left(\frac{a-1}{a}\right)^2 \sin x\,(\cos x)^{-\frac{2a+1}{a}}\left[\cos^2 x - \frac{a+1}{(a-1)^2}\right].$$

If $a = 3$ and $0 < x < \pi/2$, we have $f''(x) < 0$, $f'(x) < f'(0) = 0$, and $f(x) < f(0) = 0$ which proves (1).

If $a > 3$, then $(a+1)/(a-1)^2 < 1$ and (5) implies that there exists $\zeta \in (0, \pi/2)$ such that

$$\begin{aligned} &f''(x) > 0 &&(0 < x < \zeta),\\ &f''(\zeta) = 0, &&\\ &f''(x) < 0 &&(\zeta < x < \pi/2). \end{aligned}$$

These relations imply that the function $x \mapsto f'(x)$ is increasing from $f'(0) = 0$ to $f'(\zeta) > 0$ and then decreasing to $f'(\pi/2 - 0) = -\infty$. Hence, there is $\eta \in (0, \pi/2)$ such that

$$\begin{aligned} &f'(x) > 0 &&(0 < x < \eta),\\ &f'(\eta) = 0, &&\\ &f'(x) < 0 &&(\eta < x < \pi/2). \end{aligned}$$

From these relations we conclude that (2) holds.

Remark. If $a \leq b \leq 3$ and $0 < x < \pi/2$, then

$$\cos x < \frac{(\sin x)^a}{x^b}.$$

This is a consequence of the above result and the obvious inequality

$$(\sin x)^a \geq (\sin x)^b \qquad (a \leq b \text{ and } 0 < x < \pi/2).$$

Generalization. Let $f(x) = x - (\sin x)^p (\cos x)^q$. There exists a strictly decreasing function $p \mapsto \lambda(p)$ defined over $(-\infty, 1)$ and such that $\lambda(-\infty) = 0$, $\lambda(1-0) = -1/3$ and:

1° If $p \geq 1$ and $q \geq 0$, then $f(x) > 0$ for all $x \in (0, \pi/2)$;

2° If $p < 1$ and $q < \lambda(p)$, or $p = 1$ and $q \leq -1/3$, then $f(x) < 0$ for all $x \in (0, \pi/2)$;

3° If $p < 1$, $q \geq 0$, then there exists $x_1 \in (0, \pi/2)$ such that $f(x) < 0$ for $0 < x < x_1$, $f(x_1) = 0$, $f(x) > 0$ for $x_1 < x < \pi/2$;

4° If $p < 1$, $\lambda(p) < q < 0$, then there exist $x_1, x_2 \in (0, \pi/2)$ such that $f(x) < 0$ for $0 < x < x_1$, $f(x_1) = 0$, $f(x) > 0$ for $x_1 < x < x_2$, $f(x_2) = 0$, $f(x) < 0$ for $x_2 < x < \pi/2$;

5° If $p < 1$, $q = \lambda(p)$, then $f(x) \leq 0$ for all $x \in (0, \pi/2)$ and $f(x) = 0$ only for one value of $x = x_1 \in (0, \pi/2)$;

6° If $p > 1$, $q < 0$, or $p = 1$, $-1/3 < q < 0$, then there exists $x_1 \in (0, \pi/2)$ such that $f(x) > 0$ for $0 < x < x_1$, $f(x_1) = 0$, $f(x) < 0$ for $x_1 < x < \pi/2$.

The proof of the above result can be found in [1]. The analysis of the inequality

$$(\cos x)^c < \frac{(\sin x)^a}{x^b} \qquad (0 < x < \pi/2)$$

is given in [2].

See also [3], where the above results are checked on a ZUSE-Z 23 digital computer.

References

1. Mitrinović, D. S., and D. D. Adamović: Sur une inégalité élémentaire où interviennent des fonctions trigonométriques. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **143–155**, 23–34 (1965).

2. Mitrinović, D. S., and D. D. Adamović: Complément à l'article "Sur une inégalité élémentaire où interviennent des fonctions trigonométriques". Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **159–170**, 31–32 (1966).

3. Jovanović, S. M.: An investigation on the function $f(x) = x - (\sin x)^p (\cos x)^q$ in the interval $x \in (0, \pi/2)$ for different values of the parameters p (> 1) and q (< 0). Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **143–155**, 35–38 (1965).

3.4.19 If $x \geq \sqrt{3}$, then

$$(x+1)\cos\frac{\pi}{x+1} - x\cos\frac{\pi}{x} > 1. \tag{1}$$

Proof. Using TAYLOR's expansion of $\cos t$ near the origin, we get

$$\begin{aligned} f(x) &= (x+1)\cos\frac{\pi}{x+1} - x\cos\frac{\pi}{x} \\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{\pi^{2k}}{(2k)!\,(x+1)^{2k-1}} - \sum_{k=0}^{+\infty}(-1)^k\frac{\pi^{2k}}{(2k)!\,x^{2k-1}} \\ &= 1 + \sum_{k=1}^{+\infty}(-1)^{k-1}\frac{\pi^{2k}}{(2k)!}\left[\frac{1}{x^{2k-1}} - \frac{1}{(x+1)^{2k-1}}\right]. \end{aligned}$$

In order to prove (1), it is sufficient to show that

$$\sum_{k=1}^{+\infty}(-1)^{k-1}\frac{\pi^{2k}}{(2k)!}\left[\frac{1}{x^{2k-1}} - \frac{1}{(x+1)^{2k-1}}\right] > 0 \quad (x \geq \sqrt{3}). \tag{2}$$

The series (2) is alternating for $x \geq 0$. We can prove that

$$\frac{1}{x^{2k-1}} - \frac{1}{(x+1)^{2k-1}} > \frac{1}{x^{2k+1}} - \frac{1}{(x+1)^{2k+1}} \tag{3}$$

holds for $x \geq \sqrt{3}$ and $k = 1, 2, \ldots$ It is evident that (3) implies (2). In order to prove (3), it is sufficient to show that

$$f_k(x) = \frac{1}{x^{2k-1}} - \frac{1}{x^{2k+1}}$$

decreases for $x \geq \sqrt{3}$ $(k = 1, 2, \ldots)$.

Since

$$f_k'(x) = \frac{-(2k-1)}{x^{2k+2}}\left(x^2 - \frac{2k+1}{2k-1}\right),$$

we conclude that f_k decreases for

$$x > \sqrt{\frac{2k+1}{2k-1}} \qquad (k = 1, 2, \ldots).$$

Since

$$\sqrt{\frac{2k+1}{2k-1}} \leq \sqrt{3} \qquad (k = 1, 2, \ldots),$$

we infer that f_k for all $k = 1, 2, \ldots$ are decreasing on $[\sqrt{3}, +\infty)$.

Therefore, inequality (1) is true. ∎

3.4.20 For every $t > 0$,

$$0 < t - \frac{3\sin t}{2+\cos t} < \frac{t^5}{180},$$

$$0 < t - \frac{3\sin t}{2+\cos t}\left(1 + \frac{(1-\cos t)^2}{9(3+2\cos t)}\right) < \frac{t^7}{2100}.$$

Reference

FRAME, J. S.: Solving a right triangle without tables. Amer. Math. Monthly **51**, 36–38 (1944).

3.4.21 The following inequalities are valid:

$$\frac{(3 - x^2/10)\sin x}{2 - x^2/10 + \cos x} < x < \frac{(3 - x^2/\pi^2)\sin x}{2 - x^2/\pi^2 + \cos x} \qquad (0 < x < \pi),$$

$$\frac{(3 - x^2/20)\sin x}{1 - x^2/20 + 2\cos x} < \log\frac{1 + \sin x}{\cos x} < \frac{(3 - x^2/10)\sin x}{1 - x^2/10 + 2\cos x} \quad (0 < x < 5\pi/12).$$

Reference

Frame, J. S.: Some trigonometric, hyperbolic and elliptic approximations. Amer. Math. Monthly **61**, 623—626 (1954).

3.4.22 For all real values of x and a,

$$\frac{1}{3}(4 - \sqrt{7}) \le \frac{x^2 + x\sin a + 1}{x^2 + x\cos a + 1} \le \frac{1}{3}(4 + \sqrt{7}). \tag{1}$$

Proof. We can assume that $\sin a \neq \cos a$. Let

$$y = \frac{x^2 + x\sin a + 1}{x^2 + x\cos a + 1},$$

i.e.,

$$(y - 1)x^2 + (y\cos a - \sin a)x + (y - 1) = 0.$$

The variable y can take all values which satisfy

$$(y\cos a - \sin a)^2 - 4(y - 1)^2 \ge 0.$$

This inequality is satisfied if and only if $y_1 \le y \le y_2$, where y_1 and y_2 are roots of

$$(y\cos a - \sin a)^2 - 4(y - 1)^2 = 0.$$

These roots are found to be

$$\frac{2 - \sin a}{2 - \cos a}, \quad \frac{2 + \sin a}{2 + \cos a}.$$

In order to prove (1), it is sufficient to show that

$$\frac{1}{3}(4 - \sqrt{7}) \le \frac{2 - \sin a}{2 - \cos a} \le \frac{1}{3}(4 + \sqrt{7}), \tag{2}$$

and

$$\frac{1}{3}(4 - \sqrt{7}) \le \frac{2 + \sin a}{2 + \cos a} \le \frac{1}{3}(4 + \sqrt{7}), \tag{3}$$

for all real a.

It is sufficient to prove only (2), since (3) follows after replacing a by $a + \pi$ in (2).

Putting $f(a) = (2 - \sin a)/(2 - \cos a)$, we get

$$f'(a) = \frac{1 - 2(\cos a + \sin a)}{(2 - \cos a)^2}.$$

The roots of $f'(a) = 0$ are

$$a_1 = \alpha - \frac{\pi}{4}, \quad a_2 = -\alpha + \frac{3\pi}{4} \qquad \left(\alpha = \arcsin \frac{1}{2\sqrt{2}}\right).$$

Since

$$\sin a_1 = \frac{1-\sqrt{7}}{4}, \quad \cos a_1 = \frac{1+\sqrt{7}}{4},$$

$$\sin a_2 = \frac{1+\sqrt{7}}{4}, \quad \cos a_2 = \frac{1-\sqrt{7}}{4},$$

we find that

$$f(a_1) = \frac{1}{3}(4+\sqrt{7}), \quad f(a_2) = \frac{1}{3}(4-\sqrt{7}).$$

Hence $f(a_1) = f_{\max}$, $f(a_2) = f_{\min}$, which proves (2).

3.4.23 If $0 < \theta < t < \pi/2$, then

$$\theta \operatorname{cosec} \theta < t \operatorname{cosec} t \quad \text{and} \quad \theta \cot \theta > t \cot t.$$

From these inequalities, under the same assumptions, we get

$$1 - \frac{\sin^2\theta}{\sin^2 t} < 1 - \frac{\theta^2}{t^2} < \sec^2\theta\left(1 - \frac{\sin^2\theta}{\sin^2 t}\right).$$

If n is an odd integer and r takes the values $1, \ldots, \frac{1}{2}(n-1)$, then, for $0 < x < \pi$,

$$\Pi\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right) < \Pi\left(1 - \frac{x^2}{r^2\pi^2}\right) < \sec^{n-1}\frac{x}{n}\,\Pi\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right).$$

If $0 < \theta < \pi/2$ and $0 < t < \pi/2$, then

$$\left|1 - \frac{\sin^2\theta}{\sin^2 t}\right| \le \left|1 - \frac{\theta^2}{t^2}\right| \le \sec^2\theta\left|1 - \frac{\sin^2\theta}{\sin^2 t}\right|.$$

Reference

Durell, C. V., and A. Robson: Advanced Trigonometry. London 1948, p. 240.

3.4.24 For any real x and for any positive integer n, we have

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \le \frac{1}{n+1}, \tag{1}$$

$$\left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \le \frac{1}{n+1}. \tag{2}$$

Equality in (1) holds only for n even and $x = 0$; equality in (2) holds only for n odd and $x = 0$.

Remark. These inequalities were proposed by T. H. GRONWALL and proved by O. DUNKEL and H. S. UHLER [see Problem 339 in Amer. Math. Monthly **27**, 81–85 (1920)].

The problem of proving inequalities (1) and (2) was proposed for the first time in 1913, and then again in 1919 [see the above journal, **26**, 213 (1919)].

Before this, T. H. GRONWALL proposed to establish the identities:

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy, \tag{1'}$$

and

$$\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n}{2}\pi\right) dy. \tag{2'}$$

These identities were proved in 1913 [see Problem 331. Amer. Math. Monthly **20**, 138 (1913)].

It is interesting to note that inequalities (1) and (2) are direct consequences of (1′) and (2′) although no connection was made between Problems 331 and 339 in the aforementioned journal.

3.4.25 If

$$S_{2n-1}(x) = x - \frac{x^3}{3!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!},$$

then

$$\frac{S_{4n-1}(x_2)}{S_{4n-1}(x_1)} < \frac{\sin x_2}{\sin x_1} < \frac{S_{4n+1}(x_2)}{S_{4n+1}(x_1)} \qquad (0 < x_1 < x_2 < \sqrt{6}). \tag{1}$$

Proof. We have

$$f_n(x) = \frac{S_{2n-1}(x)}{\sin x},$$

$$\sin^2 x\, f_n'(x) = S_{2n-1}'(x)\sin x - S_{2n-1}(x)\cos x.$$

Since $S_{2n-1}''(x) = -S_{2n-3}(x)$, we obtain

$$\begin{aligned}\left(\sin^2 x\, f_n'(x)\right)' &= [S_{2n-1}(x) - S_{2n-3}(x)]\sin x\\ &= \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}\sin x.\end{aligned}$$

If n is even, we get

$$\begin{aligned}\left(\sin^2 x\, f_n'(x)\right)' &< 0 \quad \text{for} \quad 0 < x < \pi,\\ \sin^2 x\, f_n'(x) &< 0 \quad \text{for} \quad 0 < x < \pi,\\ f_n'(x) &< 0 \quad \text{for} \quad 0 < x < \pi.\end{aligned}$$

Hence

$$\frac{S_{4n-1}(x_1)}{\sin x_1} > \frac{S_{4n-1}(x_2)}{\sin x_2} \qquad (0 < x_1 < x_2 < \pi). \tag{2}$$

If $0 < x_1 < \sqrt{6}$, then

$$\frac{x_1^{4k-3}}{(4k-3)!} > \frac{x_1^{4k-1}}{(4k-1)!} \qquad (k = 1, \ldots, n)$$

so that

$$S_{4n-1}(x_1) = \sum_{k=1}^{n} \left[\frac{x_1^{4k-3}}{(4k-3)!} - \frac{x_1^{4k-1}}{(4k-1)!} \right] > 0.$$

Assuming also that $0 < x_1 < x_2 < \sqrt{6}$, we conclude that (2) can be transformed into

$$\frac{S_{4n-1}(x_2)}{S_{4n-1}(x_1)} < \frac{\sin x_2}{\sin x_1} \qquad (0 < x_1 < x_2 < \sqrt{6}).$$

The right-hand inequality in (1) can be proved in the same way.

3.4.26 If n is a positive integer, then for $|x| \leq \pi/2$,

(1) $$\operatorname{cosec}^2 x - \frac{1}{2n+1} < \sum_{k=-n}^{n} \frac{1}{(x - k\pi)^2} < \operatorname{cosec}^2 x,$$

and, for $0 < x \leq \frac{\pi}{2}$,

(2) $$\operatorname{cosec} x - \frac{x}{4n+1} < \sum_{k=-2n}^{2n} \frac{(-1)^k}{x - k\pi} < \operatorname{cosec} x + \frac{x}{4n+2}.$$

Remark. Inequalities (1) and (2) were used by E. H. Neville for evaluation of $\int_0^{+\infty} \frac{\sin x}{x}\, dx$.

Reference

Neville, E. H.: A trigonometrical inequality. Proc. Cambridge Phil. Soc. **47**, 629–632 (1951).

3.4.27 If inequalities

(1) $$t + at^3 < \tan t < t + bt^3$$

hold for each $t \in (0, \lambda)$, where $0 < \lambda < \pi/2$, then the best possible constants a and b are given by

(2) $$a = \frac{1}{3}, \quad b = \frac{\tan \lambda - \lambda}{\lambda^3}.$$

Proof. Denoting $f(t) = (\tan t - t)/t^3$, inequalities (1) become

$$a < f(t) < b \qquad (0 < t < \lambda).$$

We find that

$$f'(t) = \frac{g(t)}{t^4}, \quad g(t) = t \tan^2 t - 3 \tan t + 3t, \quad g'(t) = \frac{\sin t}{\cos^3 t} (2t - \sin 2t).$$

Hence, $g'(t) > 0$ $(0 < t < \lambda)$ which implies that $g(t) > g(0) = 0$ $(0 < t < \lambda)$. We infer that f is an increasing function on $(0, \lambda)$. Therefore, the best possible values of a and b are

$$a = f(0+) = \frac{1}{3}, \quad b = f(\lambda),$$

which is in agreement with (2).

Remark. If $\lambda = \pi/6$, then $b < 4/9$ and we have

$$t + \frac{1}{3}t^3 < \tan t < t + \frac{4}{9}t^3 \qquad (0 < t < \pi/6).$$

This proof is due to D. Ž. Djoković.

3.4.28 For $0 < \tan x < 1$, the following inequalities hold:

(1) $$\tan x - \frac{1}{3}\tan^3 x < x < \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x < \tan x;$$

(2) $$\sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}} > \tan x - \frac{1}{2}\tan^3 x;$$

(3) $$1 - \cos x = 1 - \frac{1}{\sqrt{1 + \tan^2 x}} < \frac{1}{2}\tan^2 x.$$

Remark. Inequality (1) can be obtained if x is replaced by $\tan x$ in the power series for $\arctan x$. Inequalities (2) and (3) can be obtained if x is replaced by $\tan x$ in the power series for $(1 + x^2)^{-1/2}$. In both cases alternating series are obtained. T. H. Gronwall has given inequalities (1), (2) and (3) without proof.

Reference

Gronwall, T. H.: On the influence of keyways on the stress distribution in cylindrical shafts. Trans. Amer. Math. Soc. **20**, 234—244 (1919). — See, in particular, p. 243.

3.4.29 If $0 < x < \pi/2$, then

(1) $$\frac{4}{\pi}\,\frac{x}{\pi - 2x} < \tan x.$$

Reference

Stečkin, S. B.: Some remarks on trigonometric polynomials (Russian). Uspehi Mat. Nauk (N.S.) **10**, No. 1 (63), 159—166 (1955).

3.4.30 If $0 \leq x < 1$, then

$$\frac{6(1 - x)^{1/2}}{2\sqrt{2} + (1 + x)^{1/2}} < \arccos x < \frac{\sqrt[3]{4}\,(1 - x)^{1/2}}{(1 + x)^{1/6}}.$$

These inequalities of B. C. Carlson are cited in the paper [1] mentioned in References of 3.7:1.

3.4.31 For $x > 0$, we have

$$\arcsin x > \frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} > \frac{3x}{2 + \sqrt{1-x^2}}.$$

These inequalities are due to R. E. SHAFER.

3.5 Inequalities Involving Trigonometric Polynomials

There are a large number of results on inequalities involving trigonometric polynomials. In this Section we quote some simpler cases, as well as some more general results provided that their formulation is not complicated. Many results on this topic could not be incorporated, as, for example, those contained in the following interesting papers:

ROGOSINSKI, W. W., and G. SZEGÖ: Extremum problems for nonnegative sine polynomials. Acta Sci. Math. (Szeged) **12**, Part B, 112—124 (1950).
ROGOSINSKI, W. W.: Extremum problems for polynomials and trigonometrical polynomials. J. London Math. Soc. **29**, 259—275 (1954).
ROGOSINSKI, W. W.: Linear extremum problems for real polynomials and trigonometrical polynomials. Arch. Math. (Basel) **5**, 182—190 (1954) and Corrigenda **6**, 87 (1955).
HYLTÉN-CAVALLIUS, C.: Some extremal problems for trigonometrical and complex polynomials. Math. Scand. **3**, 5—20 (1955).
MULHOLLAND, H. P.: On two extremum problems for polynomials on the unit circle. J. London Math. Soc. **31**, 191—199 (1956).

3.5.1 If $0 < x < 2\pi$, then

$$(1) \qquad -\frac{1}{2}\tan\frac{x}{4} \leq \sum_{k=1}^{n} \sin kx \leq \frac{1}{2}\cot\frac{x}{4}.$$

Proof. Since

$$\sum_{k=1}^{n} \sin kx = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

together with $-1 \leq \cos\left(n+\frac{1}{2}\right)x \leq 1$ and $0 < x < 2\pi$, we obtain (1).

Comment of D. V. SLAVIĆ. The statement in Elem. Math. **10**, 69—70 (1955), that (1) holds for any $x \neq 2m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ is not true.

3.5.2 If k and n are positive integers and if x is a real number such that $x \neq 2m\pi$ for $m = 0, \pm 1, \pm 2, \ldots$, then

$$(1) \qquad \left|\frac{1}{2} + \sum_{k=1}^{n} \cos kx\right| \leq \left|\frac{1}{2}\operatorname{cosec}\frac{x}{2}\right|.$$

Proof. Since

$$\sum_{k=1}^{n} \cos kx = -\frac{1}{2} + \frac{1}{2} \operatorname{cosec} \frac{x}{2} \sin\left(n + \frac{1}{2}\right) x,$$

together with $-1 \leq \sin(n + 1/2)\, x \leq 1$ and $x \neq 2m\pi$, we obtain (1).

3.5.3 Let k and m be nonnegative integers, n a natural number and let D_k denote DIRICHLET's kernel defined by

$$D_k(x) = \frac{1}{2} + \cos x + \cdots + \cos kx = \frac{\sin\left(k + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}.$$

Then

$$\sum_{r=0}^{m} D_k\left(\frac{2\pi}{n} r\right) > 0 \quad \text{for} \quad k, m = 0, 1, \ldots \quad \text{and} \quad n = 1, 2, \ldots$$

If $0 \leq k < n$ and $1 \leq m < n$, from the above inequality it follows

$$\sum_{r=1}^{m} D_k\left(\frac{2\pi}{n} r\right) < \frac{n}{2} \quad \text{and} \quad \sum_{r=1}^{m} D_k\left(\frac{2\pi}{n} r\right) < \frac{n - k + m}{2}.$$

Reference

MAKAI, E.: A property of Dirichlet's kernel. Studia Sci. Math. Hung. **1**, **11–16** (**1966**).

3.5.4 Let n and $m + 1$ be natural numbers. If $0 < x < 2\pi$, then

$$\left| \sum_{k=m+1}^{m+n} e^{ikx} \right| \leq \frac{1}{\sin \frac{x}{2}}. \tag{1}$$

Proof. Since

$$\left| \sum_{k=m+1}^{m+n} e^{ikx} \right| = \left| e^{i(m+1)x} \frac{e^{inx} - 1}{e^{ix} - 1} \right| = \left| \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \right|,$$

inequality (1) is evident.

See also 3.5.37.

3.5.5 For all natural numbers n and all real x,

$$\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| < \int_0^{\pi} \frac{\sin x}{x}\, dx = 1.8519\ldots < \frac{\pi}{2} + 1.$$

References

JACKSON, D.: Über eine trigonometrische Summe. Rend. Circ. Mat. Palermo **32**, **257–262** (**1911**).

GRONWALL, T. H.: Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx$. Math. Ann. **72**, 228–243 (1912).

3.5.6 If $0 < x < \pi$, then

$$0 < \sum_{k=1}^{n} \frac{\sin kx}{k} < \pi - x.$$

References

TURÁN, P.: Über die Partialsummen der Fourierreiche. J. London Math. Soc. **13**, 278–282 (1938).

HYLTÉN-CAVALLIUS, C.: Geometrical methods applied to trigonometrical sums. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar **21**, No. **1**, **19** pp. (1950).

3.5.7 For all $n = 1, 2, \ldots$ and for real x

$$\left| \sum_{k=1}^{n} \frac{(-1)^k}{k} \sin kx \right| \leq \sqrt{2}\, |x|. \tag{1}$$

Proof. First let $x \geq \pi/2$. Then the right-hand side is > 2, whereas the left-hand side is (with $x = \pi + y$)

$$\left| \sum_{k=1}^{n} \frac{(-1)^k}{k} \sin kx \right| = \left| \sum_{k=1}^{n} \frac{\sin ky}{k} \right| < \int_0^{\pi} \frac{\sin u}{u}\, du < 2.$$

If $x \mapsto |g(x)|$ denotes the function on the left, we have, for $0 < x < \pi/2$,

$$|g'(x)| = \left| \sum_{k=1}^{n} (-1)^k \cos kx \right| = \left| \sum_{k=1}^{n} \cos ky \right| \leq \frac{1}{\left|\sin \frac{y}{2}\right|} \leq \sqrt{2};$$

since $-\pi < y < -\pi/2$. This yields $|g(x)| \leq \sqrt{2}\,|x|$ in $0 < x < \pi/2$, which completes the proof of (1).

Remark. Compare the above result with 3.5.6.

Reference

GAIER, D., and J. TODD: On the rate of convergence of optimal ADI processes. Numer. Math. **9**, 452–459 (1967).

3.5.8 Let n be a positive integer and let $0 < x < \pi/(n+1)$. For

$$T(x) = \sin x - \frac{\sin 2x}{2} + \cdots + (-1)^{n+1} \frac{\sin nx}{n} - \frac{x}{2}$$

we have

$$T(x) > 0 \quad \text{if } n \text{ is odd,} \quad \text{and} \quad T(x) < 0 \quad \text{if } n \text{ is even.}$$

B. SZ.-NAGY proved the following general theorem: Let $a_{n+1}, a_{n+2}, \ldots$ be a sequence of real numbers with the following properties:

$$a_{n+1} > 0, \;\; a_k \geq 0, \;\; a_k - a_{k+1} \geq 0, \;\; a_k - 2a_{k+1} + a_{k+2} \geq 0$$

for $k \geq n + 1$ and $\lim\limits_{k \to +\infty} a_k = 0$. Then

$$(-1)^n \sum_{k=n+1}^{+\infty} a_k \sin kx > 0$$

on the interval $\left(\dfrac{n\pi}{n+1}, \pi\right)$.

References

SZÁCZ, G., L. GEHÉR, I. KOVÁCS and L. PINTÉR: Contests in Higher Mathematics. Budapest 1968, p. 19 and pp. 175–176.

SZ.-NAGY, B.: Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. I: Periodischer Fall. Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig **90**, 103–134 (1938). — See, in particular, p. 131.

3.5.9 Let $m + 1$ and n be natural numbers. If $0 < x < 2\pi$, then

$$\left| \sum_{k=m+1}^{m+n} \frac{\sin kx}{k} \right| \leq \frac{1}{(m+1)\sin \dfrac{x}{2}}. \tag{1}$$

Proof. Starting from 3.5.4 we get

$$\left| \sum_{k=m+1}^{m+n} \sin kx \right| = \left| \operatorname{Im} \sum_{k=m+1}^{m+n} e^{ikx} \right| \leq \frac{1}{\sin \dfrac{x}{2}}, \tag{2}$$

and using ABEL's identity

$$\sum_{k=m+1}^{m+n} u_k v_k = (u_{m+1} - u_{m+2})\, v_{m+1} + (u_{m+2} - u_{m+3})\,(v_{m+1} + v_{m+2})$$
$$+ \cdots + u_{m+n}(v_{m+1} + \cdots + v_{m+n}),$$

we obtain

$$\left| \sum_{k=m+1}^{m+n} \frac{\sin kx}{k} \right| = \left| \left(\frac{1}{m+1} - \frac{1}{m+2}\right) \sin(m+1)\, x \right.$$
$$+ \left(\frac{1}{m+2} - \frac{1}{m+3}\right) [\sin(m+1)\, x + \sin(m+2)\, x]$$
$$\left. + \cdots + \frac{1}{m+n} [\sin(m+1)\, x + \cdots + \sin(m+n)\, x] \right|.$$

Using (2) we infer that

$$\left|\sum_{k=m+1}^{m+n} \frac{\sin kx}{k}\right| \leq \left|\left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \left(\frac{1}{m+2} - \frac{1}{m+3}\right) + \cdots + \frac{1}{m+n}\right|$$

$$\times \frac{1}{\sin \frac{x}{2}} = \frac{1}{(m+1)\sin \frac{x}{2}},$$

which proves (1).

3.5.10 If $-\pi < t < \pi$ and $m = 0, 1, \ldots, n = 1, 2, \ldots,$ then

$$\sum_{k=m+1}^{m+n} (-1)^{k+1} \frac{\sin kt}{k} \leq \frac{2}{(m+1)\cos(t/2)}. \tag{1}$$

Reference

Ostrowski **2**, p. 109.

Remark. Inequality (1) can be obtained from (1) in 3.5.9 replacing x by $\pi - t$.

3.5.11 If $0 < x < \pi$, then

$$\sum_{k=1}^{n} \frac{\sin kx}{k} > 4 \sin^2 \frac{x}{2} \left(\cot \frac{x}{2} - \frac{\pi - x}{2}\right) \quad (n > 1).$$

Reference

Turán, P.: On a trigonometrical sum. Ann. Soc. Polon. Math. **25**, 155–161 (1952).

3.5.12 If $0 \leq x \leq \pi$, then

$$\sum_{k=1}^{n} \frac{\cos kx}{k} \leq -\log \sin \frac{x}{2} + \frac{\pi - x}{2}.$$

Remark. C. Hyltén-Cavallius has obtained the above inequality, as well as some others, using geometrical methods.

Reference

Hyltén-Cavallius, C.: Geometrical methods applied to trigonometrical sums. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar **21**, No. 1, 19 pp. (1950).

3.5.13 If $0 \leq x \leq 2\pi/3$, then

$$\sum_{k=0}^{n} \binom{n+2-k}{2} k \sin kx > 0.$$

Remark. This inequality was first proved by G. Szegö [1] but only for the interval $0 \leq x \leq \gamma$, where $\sin^2(\gamma/2) = 0.7$ and $\pi/2 < \gamma < \pi$. M. Schweitzer [2] later proved that γ can be replaced by $2\pi/3$ and that γ cannot be greater than $2\pi/3$.

References

1. SZEGÖ, G.: Power series with multiply monotonic sequences of coefficients. Duke Math. J. **8**, 559–564 (1941).
2. SCHWEITZER, M.: The partial sums of second order of the geometrical series. Duke Math. J. **18**, 527–533 (1951).

3.5.14 For all real x,

$$\sum_{k=1}^{n} (n-k+1)\,|\sin kx| \le \frac{(n+1)^2}{\pi}.$$

Reference

TURECKIĬ, A. H.: On a function deviating least from zero (Russian). Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **16**, 41–43 (1954).

3.5.15 In 1913 W. H. YOUNG [1] proved that the cosine polynomials

$$T_n^\alpha(t) = \frac{1}{1+\alpha} + \frac{\cos t}{1+\alpha} + \cdots + \frac{\cos nt}{n+\alpha}$$

are nonnegative whenever $-1 < \alpha \le 0$.

In 1928 W. W. ROGOSINSKI and G. SZEGÖ [2] extended this result to $-1 < \alpha \le 1$. They also showed that there is a number A, $1 \le A \le 2(1+\sqrt{2})$, such that the polynomials $T_n^\alpha(t)$, for $n = 0, 1, \ldots$, are nonnegative for $-1 < \alpha \le A$, while this is not the case for $\alpha > A$.

In 1969 G. GASPER [3] proved the following result:

Let A be the positive root of the equation

$$9\alpha^7 + 55\alpha^6 - 14\alpha^5 - 948\alpha^4 - 3247\alpha^3 - 5013\alpha^2 - 3780\alpha - 1134 = 0.$$

If $-1 < \alpha \le A$, then $T_n^\alpha(t) \ge 0$ for $n = 0, 1, \ldots$ However, if $\alpha > A$ then $T_3^\alpha(t) < 0$ for some t.

A simple computation yields $A = 4.5678018\ldots$

G. GASPER also showed that: 1° $T_k^\alpha(t) \ge 0$ for $k = 0, 1, \ldots, n$ implies $T_n^\beta(t) \ge 0$ if $\alpha > \beta > -1$; and 2° $T_n^\alpha(t) > 0$ for $n \ge 4$ and $\alpha = 4.57$.

As an extension of the above results, G. GASPER also proved:

If $a_0 \ge a_1 \ge \cdots \ge a_n \ge 0$, then

$$\frac{a_0}{1+A} + \frac{a_1 \cos t}{1+A} + \cdots + \frac{a_n \cos nt}{n+A} \ge 0,$$

and if $-1 < \alpha \le A$, then for all real $t_1, \ldots, t_m$

$$\frac{1}{1+\alpha} + \sum_{k=1}^{n} \frac{1}{k+\alpha} \prod_{j=1}^{m} \cos kt_j \ge 0.$$

References

1. YOUNG, W. H.: On a certain series of Fourier. Proc. London Math. Soc. **(2) 11**, 357–366 (1913).

2. ROGOSINSKI, W. W., and G. SZEGÖ: Über die Abschnitte von Potenzreihen, die in einem Kreise beschränkt bleiben. Math. Z. **28**, 73—94 (1928).
3. GASPER, G.: Nonnegative sums of cosine, ultraspherical and Jacobi polynomials. J. Math. Anal. Appl. **26**, 60—68 (1969).

3.5.16 If $b_0, \ldots, b_n$ is a nonincreasing sequence, then, for all real p and q and $0 < x < \pi$,

$$-b_0 \frac{\sin^2\left(q - \frac{p}{2}\right)\frac{x}{2}}{\sin\frac{px}{2}} \leq \sum_{k=0}^{n} b_k \sin(kp + q)\, x \leq b_0 \frac{\cos^2\left(q - \frac{p}{2}\right)\frac{x}{2}}{\sin\frac{px}{2}}.$$

Reference

KARAMATA, J., and M. TOMIĆ: Considérations géométriques relatives aux polynomes et séries trigonométriques. Acad. Serbe Sci. Publ. Inst. Math. **2**, 157—175 (1948).

3.5.17 If $a_0, \ldots, a_n$ is a positive, nonincreasing and convex sequence, then for $0 < x < 2\pi$,

$$0 \leq \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx \leq \frac{a_0 - a_1}{2} \operatorname{cosec}^2 \frac{x}{2}.$$

See the reference in **3.5.16**.

3.5.18 Let $b = (b_1, \ldots, b_n)$ be a sequence of positive real numbers. If this sequence is monotonely decreasing and convex, then for $0 \leq x \leq \pi$,

$$b_1 \sin x + \cdots + b_{n-1} \sin(n-1)\, x + \frac{b_n}{2} \sin nx \geq 0.$$

If the sequence of nonnegative real numbers $a = (a_0, \ldots, a_n)$ is monotonely decreasing, convex, and if $a_{n-1} \geq 2a_n$, then for all real x,

$$\frac{1}{2} a_0 + a_1 \cos x + \cdots + a_n \cos nx \geq 0.$$

Reference

FEJÉR, L.: Einige Sätze, die sich auf das Vorzeichen einer ganzen rationalen Funktion beziehen; nebst Anwendungen dieser Sätze auf die Abschnitte und Abschnittsmittelwerte von ebenen und räumlichen harmonischen Entwicklungen und von beschränkten Potenzreihen. Monatsh. Math. **35**, 305—344 (1928).

This article contains numerous interesting inequalities concerning trigonometric polynomials together with the literature on that subject.

3.5.19 Let $\sum_{k=1}^{n} b_k \sin(2k-1)\,\theta \geq 0$ for $0 \leq \theta \leq \pi$. Then

$$\sum_{k=1}^{n} \frac{b_k}{k} \sin k\varphi > 0 \quad \text{for} \quad 0 < \varphi < \pi,$$

unless all $b_k = 0$.

Proof. We have

$$\frac{d}{dt}\left(\frac{\sin \alpha t}{\alpha (\sin t)^{\alpha}}\right) = -\frac{\sin (\alpha - 1)\, t}{(\sin t)^{\alpha+1}}.$$

Letting $\alpha = 2k$ and $t = \varphi/2$, we obtain

$$\frac{\sin k\varphi}{k} = 2\int_{\varphi/2}^{\pi/2} \left(\frac{\sin \varphi/2}{\sin \theta}\right)^{2k} \frac{\sin (2k-1)\,\theta}{\sin \theta}\, d\theta.$$

Thus

$$\sum_{k=1}^{n} \frac{b_k \sin k\varphi}{k} = 2\int_{\varphi/2}^{\pi/2} \sum_{k=1}^{n} b_k \left(\frac{\sin \varphi/2}{\sin \theta}\right)^{2k} \sin (2k-1)\,\theta\, \frac{d\theta}{\sin \theta}.$$

But $\sum_{k=1}^{n} b_k r^{2k-1} \sin(2k-1)\,\theta > 0$ for $0 < r < 1$ if $\sum_{k=1}^{n} b_k \sin(2k-1)\,\theta \geq 0$ and not all b_k are zero.

This completes the proof.

The above result is due to P. Turán [1] and the cited proof to R. Askey, J. Fitch and G. Gasper [2].

See 3.5.20.

References

1. Turán, P.: On a trigonometrical sum. Ann. Soc. Polon. Math. **25**, 155–161 (1952).
2. Askey, R., J. Fitch and G. Gasper: On a positive trigonometric sum. Proc. Amer. Math. Soc. **19**, 1507 (1968).

3.5.20 If not all real numbers $a_0, \ldots, a_n$ are equal to zero and if

$$\sum_{k=0}^{n} a_k = 0, \qquad \sum_{k=0}^{n} a_k \cos kx \geq 0 \quad (-\pi \leq x \leq \pi),$$

then

$$\sum_{k=1}^{n} \frac{a_0 + \cdots + a_{k-1}}{k} \sin kx > 0 \quad (0 < x < \pi).$$

References

Turán, P.: On a trigonometrical sum. Ann. Soc. Polon. Math. **25**, 155–161 (1952).

Hyltén-Cavallius, C.: A positive trigonometrical kernel. C. R. 12e Congrès Math. Scandinaves, Lund 1953, pp. 90–94 (1954).

3.5.21 If $0 < a \leq \pi$, then

$$\max_{-a \leq t \leq a} \left| \cos nt + \sum_{k=0}^{n-1} a_k \cos kt \right| \geq \left(\sin \frac{a}{2}\right)^{2n}.$$

This result is due to S. B. Bernstein. See Timan, A. F.: Theory of Approximation of Functions of a Real Variable. Oxford-London-New York-Paris 1963, p. 90.

3.5.22 Denoting by E the class of trigonometric polynomials of the form

$$C(x) = c_0 + c_1 \cos x + \cdots + c_n \cos nx,$$

where $c_0 \geq c_1 \geq \cdots \geq c_n > 0$, then

$$\left(1 - \frac{2}{\pi}\right)\frac{1}{n+1} \leq \min_{C\in E} \frac{\max\limits_{\pi/2\leq x\leq\pi} |C(x)|}{\max\limits_{0\leq x\leq 2\pi} |C(x)|} \leq \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\frac{1}{n+1}.$$

Reference

SZÁCZ, G., L. GEHÉR, I. KOVÁCS and L. PINTÉR: Contests in Higher Mathematics. Budapest 1968, p. 20 and pp. 177—178.

3.5.23 Let $a_0, a_1, \ldots, a_n$ and t be real numbers. If

$$a_0 \geq a_1 \geq \cdots \geq a_n > 0, \tag{1}$$

and

$$a_{2k} \leq \frac{2k-1}{2k} a_{2k-1} \qquad (1 \leq k \leq n/2), \tag{2}$$

then

$$\sum_{k=1}^{n} a_k \sin kt > 0, \qquad \sum_{k=0}^{n} a_k \cos kt > 0 \qquad (0 < t < \pi). \tag{3}$$

These inequalities are due to L. VIETORIS [1].

Putting $a_0 = 1$, $a_k = 1/k$ ($k = 1, \ldots, n$), inequalities (3) become

$$\sum_{k=1}^{n} \frac{1}{k} \sin kt > 0 \qquad (0 < t < \pi), \tag{4}$$

$$1 + \sum_{k=1}^{n} \frac{1}{k} \cos kt > 0 \qquad (0 < t < \pi). \tag{5}$$

Since in this case conditions (1) and (2) are fulfilled, the truth of (4) and (5) is an immediate consequence of (3). Inequality (4) is known as the FEJÉR-JACKSON inequality and (5) as the W. H. YOUNG inequality. There exist also various proofs of (4) which will not be mentioned here.

Generalization. D. Ž. DJOKOVIĆ [2] proved the following result:

If t_ν ($\nu = 1, \ldots, m$) satisfy

$$t_\nu > 0 \quad (\nu = 1, \ldots, m), \qquad \sum_{\nu=1}^{m} t_\nu < \pi, \tag{6}$$

then

$$\sum_{k=1}^{n} \left(\prod_{\nu=1}^{m} \frac{\sin kt_\nu}{k}\right) > 0 \quad (n = 1, 2, \ldots). \tag{7}$$

For $m = 1$, (7) reduces to the FEJÉR-JACKSON inequality (4).

The proof of (7) is based on (4) and proceeds by induction on m. Note that the function F defined by

$$F(x) = \sum_{k=1}^{n} \left(\prod_{\nu=1}^{m-1} \frac{\sin kt_\nu}{k} \right) \left(\frac{\sin kx}{k} \right)^2$$

is increasing for $0 \leq x \leq \frac{1}{2}(\pi - t_1 - \cdots - t_{m-1})$.

L. Vietoris communicated to us the following result:

If (1) and (2) hold, then

$$\sum_{k=1}^{n} \left(ka_k \prod_{\nu=1}^{m} \frac{\sin kt_\nu}{k} \right) > 0$$

for $0 < t_\nu < \pi$ $(\nu = 1, \ldots, m)$ and $\sum_{\nu=1}^{m} t_\nu \leq \pi$.

References

1. Vietoris, L.: Über das Vorzeichen gewisser trigonometrischer Summen. Sitzungsber. Öst. Akad. Wiss. **167**, 125–135 (1958), and Anzeiger Öst. Akad. Wiss. 1959, pp. 192–193.
2. Djoković, D. Ž.: Sur une généralisation de l'inégalité de Fejér-Jackson. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **35–37**, 1–4 (1960).

3.5.24 Let $T_n(x) = \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ with real a_k, b_k. If

$$\max_{1 \leq k \leq n} (|a_k|, |b_k|) = 1 \quad \text{and} \quad \sum_{k=1}^{n} (a_k^2 + b_k^2) = An,$$

then there exists $c > 0$ which depends only on A, such that $\lim_{A \to 0} c(A) = 0$, and

$$\max_{0 \leq x < 2\pi} |T_n(x)| > \frac{1 + c(A)}{\sqrt{2}} \left(\sum_{k=1}^{n} (a_k^2 + b_k^2) \right)^{1/2}.$$

Reference

Erdös, P.: An inequality for the maximum of trigonometric polynomials. Ann. Polon. Math. **12**, 151–154 (1962).

3.5.25 If in the interval $-a \leq x \leq +a$ $(0 < a < \pi)$ the trigonometric polynomial

$$T_n(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)$$

satisfies the inequality $\max_{-a \leq x \leq a} |T_n(x)| \leq 1$, then for any x

$$|T_n(x)| \leq \frac{1}{2} \left(\tan^{2n} \frac{a}{4} + \tan^{-2n} \frac{a}{4} \right).$$

Reference

Gončarov, V. L.: Theory of Interpolation and Approximation of Functions (Russian). 2nd ed., Moscow 1954, pp. 231–232.

3.5.26 Let $a_1, \ldots, a_n$ be real numbers and

$$(1) \qquad S = \sum_{k=1}^{n} \exp(2a_k i),$$

$$(2) \qquad S' = \frac{1}{2}\exp(2a_1 i) + \frac{1}{2}\exp(2a_n i) + \sum_{k=2}^{n-1}\exp(2a_k i).$$

If

$$(3) \qquad 0 < \theta \leq a_2 - a_1 \leq \cdots \leq a_n - a_{n-1} \leq \varphi < \pi,$$

then

$$(4) \qquad 2\,|S| \leq \cot\frac{\theta}{2} + \tan\frac{\varphi}{2},$$

$$(5) \qquad 2\left|S - \frac{i\exp(2a_1 - \theta)\,i}{2\sin\theta}\right| \leq \cot\theta + \tan\frac{\varphi}{2}.$$

If $\Delta a_k = a_{k+1} - a_k$, $\Delta^2 a_k = \Delta(\Delta a_k)$, $\ldots$, and

$$0 < \theta \leq \Delta a_k \leq \pi/4 \quad (k = 1, \ldots, n-1),$$

$$\Delta^2 a_k \geq 0 \quad (k = 1, \ldots, n-2),$$

$$\Delta^3 a_k \geq 0 \quad (k = 1, \ldots, n-3),$$

then

$$(6) \qquad |S| \leq \frac{1}{\sin\theta}.$$

If (3) holds and $\varphi \leq \pi/2$, then

$$(7) \qquad |S'| \leq \cot\theta.$$

Proof. The identity

$$2\sum_{k=1}^{n}\lambda_k = \lambda_1\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+1\right) + \lambda_n\left(1 - \frac{\lambda_{n-1}+\lambda_n}{\lambda_{n-1}-\lambda_n}\right)$$
$$+ \sum_{k=2}^{n-1}\lambda_k\left(\frac{\lambda_k+\lambda_{k+1}}{\lambda_k-\lambda_{k+1}} - \frac{\lambda_{k-1}+\lambda_k}{\lambda_{k-1}-\lambda_k}\right)$$

after the substitution $\lambda_k = \exp(2a_k i)$ yields

$$2S = e^{2a_1 i}\big(1 + i\cot(a_2 - a_1)\big) + e^{2a_n i}\big(1 - i\cot(a_n - a_{n-1})\big)$$
$$+ \sum_{k=2}^{n-1} i e^{2a_k i}\left[\cot(a_{k+1} - a_k) - \cot(a_k - a_{k-1})\right].$$

Using (3) and the fact that $\cot x$ decreases in $(0, \pi)$, we infer

(8) $$2\,|S| \le \frac{1}{\sin(a_2 - a_1)} + \frac{1}{\sin(a_n - a_{n-1})} + \cot(a_2 - a_1) - \cot(a_n - a_{n-1}) \le \cot\frac{a_2 - a_1}{2} + \tan\frac{a_n - a_{n-1}}{2}$$

and (4) follows immediately.

Inequalities (5), (6), and (7) can also be proved starting from (8).

Remark. For the complete proof and bibliographical references see the following paper by L. J. Mordell: On the Kusmin-Landau inequality for exponential sums. Acta Arith. **4**, 3–9 (1958), and the following book by J. F. Koksma: Diophantische Approximationen. Berlin 1936.

3.5.27 Let T_n be a trigonometric polynomial of order n, defined by

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

and let $x_1, \ldots, x_m$ $(m \ge 2)$ be distinct points in $(-\pi, \pi)$ such that

$$\min_{j \neq k} |x_j - x_k| = 2\delta.$$

Then we have the following inequalities due to H. Davenport and H. Halberstam [1]:

(1) $$\sum_{k=1}^{m} |T_n(x_k)|^2 \le 4.4 \max\left(n, \frac{\pi}{2\delta}\right) \sum_{k=-n}^{n} |c_k|^2;$$

and the following due to M. Izumi and S. Izumi [2]:

(2) $$\sum_{k=1}^{n} |T_n(x_k)|^2 \le A \sum_{k=-n}^{n} |c_k|^2$$

for small δ, where $A \le 2.34\left(N + \frac{\pi}{\delta}\right)$ or $A \le 3.13\left(N + \frac{\pi}{2\delta}\right)$.

Neither of inequalities (1) and (2) implies the other.

If $1/p + 1/q = 1$ $(p \ge 2)$, then under the same conditions, we have

$$\sum_{k=1}^{m} |T_n(x_k)|^p \le A\sqrt{p}\max\left(n, \frac{2\pi}{\delta}\right)\left(\sum_{k=-n}^{n} |c_k|^q\right)^{p/q},$$

where A is an absolute constant (see [1]) and

$$\sum_{k=1}^{n} |T_n(x_k)|^p \le A'(1 + \varepsilon)\left(N + \frac{\pi}{\delta}\right)\left(\sum_{k=-n}^{n} |c_k|^q\right)^{p/q}$$

for any $\varepsilon > 0$ and sufficiently small δ, where

$$A' = \frac{2^{p-2}}{\pi^p (q+1)^{p-1}} \left(\int_0^{+\infty} \frac{|\sin t|^q}{t^q}\,dt\right)^{p-1} \left(\int_0^{\pi/2} \frac{\sin^2 t}{t^2}\,dt\right)^{-p}.$$

References

1. DAVENPORT, H., and H. HALBERSTAM: The values of a trigonometric polynomial at well spaced points. Mathematika **13**, 91–96 (1966).
2. IZUMI, M., and S. IZUMI: On a theorem concerning trigonometrical polynomials. Proc. Japan Acad. **43**, 71–76 (1967).

3.5.28 If T_n is a trigonometric polynomial of order n, then for $0 < \delta < 2\pi/n$

$$\left|\frac{d^r}{dx^r} T_n(x)\right| \le \left(\frac{n}{2} \operatorname{cosec} \frac{n\delta}{2}\right)^r \max_x \left|\sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} T_n(x + k\delta)\right|,$$

with equality if and only if

$$T_n(x) = a \cos nx + b \sin nx + c.$$

Reference

STEČKIN, S. B.: A generalization of some inequalities of S. N. Bernštein (Russian). Doklady Akad. Nauk SSSR (N.S.) **60**, 1511–1514 (1948).
See also the review by R. P. BOAS in Math. Reviews **9**, 579–580 (1948).

3.5.29 Let T_n be a trigonometric polynomial of order n with $T(0) = 0$ and

$$\left|T_n\left(\frac{(2k+1)\pi}{2n}\right)\right| \le 1 \quad (k = 0, 1, \ldots, 2n-1).$$

Then, for $|x| \le \pi/(2n)$,

$$|T_n(x)| \le |\sin nx|. \tag{1}$$

There is equality in (1) for some x (and hence for all x) if and only if $T_n(x)$ is a constant multiple of $\sin nx$ whose modulus is 1. If $T_n(x)$ is real, the hypothesis $T_n(0) = 0$ can be replaced by $T_n(0) \le 0$. If $\pi/(2n) < x_0 \le \pi$, there is a trigonometric polynomial T of order not exceeding n with $|T(x)| \le 1$, $T(0) = 0$ and $T(x_0) = 1$.

Reference

BOAS, R. P.: Inequalities for polynomials with a prescribed zero. In: Studies in Mathematical Analysis and Related Topics. Stanford 1962, pp. 42–47.

3.5.30 If m_k are natural numbers and $0 < \delta \le \pi$, then for all real t

$$\max_{-\pi \le x \le \pi} \left|\sum_{k=1}^{n} a_k e^{i m_k x}\right| \le \left(\frac{48\pi}{\delta}\right)^n \max_{t-\delta \le x \le t+\delta} \left|\sum_{k=1}^{n} a_k e^{i m_k x}\right|.$$

This inequality is due to P. TURÁN (see A. DINGHAS: Vorlesungen über Funktionentheorie. Berlin-Göttingen-Heidelberg 1961, p. 128).

3.5.31 If $C_n(t) = \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kt$, then

$$\left|\sum_{k=0}^{n} \frac{a_k}{n-k+1}\right| \le A \int_0^{\pi} |C_n(t)|\, dt,$$

where

$$\frac{\pi}{2} \le A < \int_0^{\pi} \frac{\sin t}{t}\, dt = 1.85\ldots$$

Under the additional requirement that the a_k's have the same sign, the exact value of A is $\pi/2$.

References

HILLE, E., and J. TAMARKIN: On the summability of Fourier series I. Trans. Amer. Math. Soc. **34**, 757–783 (1932).

SIDON, S.: Über Fourier-Koeffizienten. J. London Math. Soc. **13**, 181–183 (1938).

STEČKIN, S. B.: Some remarks on trigonometric polynomials (Russian). Uspehi Mat. Nauk (N.S.) **10**, No. 1 (63), 159–166 (1955).

Remark. For some refinements of the cited results, see: Ref. Žurnal Matematika 1956, No. 9, Review 6512 by A. F. TIMAN.

3.5.32 Let $1 \le q \le +\infty$. Then, for any trigonometric polynomial T_n of order n,

$$\left(\int_0^{2\pi} |T_n^{(r)}(x)|^q\, dx\right)^{1/q} \le n^r \left(\int_0^{2\pi} |T_n(x)|^q\, dx\right)^{1/q}.$$

Reference

NIKOL'SKIĬ, S. M.: Generalization of a proposition of S. N. Bernštein (Russian). Doklady Akad. Nauk SSSR (N.S.) **60**, 1507–1510 (1948).

3.5.33 Let $P(z) = z^n \pm z^{n-1} \pm \cdots \pm 1,\ n > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|\, d\theta < \sqrt{n + 0.97}.$$

Remark. D. J. NEWMAN observes: "To prove, however, that

$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|\, d\theta < \sqrt{n}$$

for n large would call for a completely new approach. It has even been conjectured that

$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|\, d\theta \le c\sqrt{n+1},$$

for $0 < c < 1$, but we are far from this result".

This result is related to 3.5.24.

Reference

NEWMAN, D. J.: Norms of polynomials. Amer. Math. Monthly **67**, 778–779 (1960).

3.5.34 If P is an algebraic polynomial of order n and $p \geq 1$, then

$$\int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta \leq \sqrt{\pi} \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p+1}{2}\right)} n^p \int_0^{2\pi} |\operatorname{Re} P(e^{i\theta})|^p \, d\theta,$$

with equality if and only if $P(z) = Az^n$.

Reference

ZYGMUND, A.: Two notes on inequalities. J. Math. and Phys. **21**, 117–123 (1942).

3.5.35 Let

$$M(f) = \frac{1}{2\pi} \int_0^{2\pi} |f(x)| \, dx, \qquad M_n(f) = \frac{1}{2n+1} \sum_{k=1}^{2n+1} \left| f\left(\frac{2k\pi}{2n+1}\right)\right|.$$

If $T(x)$ is a trigonometric polynomial of degree n and $\overline{T}(x)$ is its conjugate trigonometric polynomial, then

$$\begin{aligned} M_n(T^p) &\leq A^p M(T^p) && (1 \leq p \leq +\infty), \\ M(T^p) &\leq B^p M_n(T^p) && (1 < p < +\infty), \\ M_n(\overline{T}^p) &\leq C^p M(T^p) && (1 < p < +\infty), \\ M(\overline{T}^p) &\leq D^p M_n(T^p) && (1 < p < +\infty), \\ M(T^p) \text{ and } M(\overline{T}^p) &\leq E^p M_n(T)^p && (0 < p < 1), \end{aligned}$$

where A is an absolute constant, while B, C, D, E depend on p.

Reference

MARCINKIEWICZ, J., and A. ZYGMUND: Mean values of trigonometric polynomials. Fund. Math. **28**, 131–166 (1936).

3.5.36 Let T be a real trigonometric polynomial defined by

$$T(t) = \sum_{k=-n}^{n} a_k e^{kit} \quad \text{with} \quad a_{-k} = \overline{a_k},$$

and let $I_n = \int_0^{2\pi} |T(t)| \, dt$.

A number of papers (see [1], [2], [3]) were devoted to determining the best possible estimate of

$$\text{(1)} \qquad \max \frac{\lambda_0 |a_0| + \lambda_k |a_k|}{I_n} \quad \text{for} \quad 0 < k \leq n,$$

where λ_0, λ_k are given nonnegative numbers.

In the cited papers the best possible estimate was found only for $k > n/2$ $(\lambda_0 \neq 0)$ or $k > n/3$ $(\lambda_0 = 0)$.

YA. L. GERONIMUS [4] showed that the problem in question is only a special case of a result of his own (see [5]), in which he established the equivalence of finding

$$\max \frac{\sum_{r=-n}^{n} a_r c_r}{I_n},$$

where $(c_1, \ldots, c_n)$ is a given sequence of complex numbers, with a result of N. I. AHIEZER and M. G. KREĬN [6].

YA. L. GERONIMUS proved that

$$|a_0| + \lambda\, |a_k| \leq \frac{1}{4\alpha} I_n,$$

where λ is an arbitrary positive number, α is the least positive root of

$$\begin{vmatrix} m_0 & m_1 & \cdots & m_p \\ m_{-1} & m_0 & & m_{p-1} \\ \vdots & & & \\ m_{-p} & m_{-p+1} & & m_0 \end{vmatrix} = 0$$

and where $m_0 = 2\cos\alpha$, $m_s = \frac{(2\alpha\lambda)^s}{s!} e^{i\alpha}$, $m_{-s} = \overline{m}_s$ for $s = 1, \ldots, p$.

This estimate is the best possible.

The method used by GERONIMUS in [5] is essentially analogous to the method used by BOAS in [3], but BOAS did not make use of the results [6], without which, as can be seen from [4], the best possible estimate of (1) cannot be obtained.

References

1. VAN DER CORPUT, J. G., and C. VISSER: Inequalities concerning polynomials and trigonometric polynomials. Indagationes Math. **8**, 238–247 (1946).
2. BOAS, R. P.: Inequalities for the coefficients of trigonometric polynomials. Indagationes Math. **9**, 298–301 (1947).
3. BOAS, R. P.: Inequalities for the coefficients of trigonometric polynomials II. Indagationes Math. **9**, 369–372 (1947).
4. GERONIMUS, YA. L.: Refinement of estimates of van der Corput, Visser, Fejes and Boas for the coefficients of trigonometric polynomials (Russian). Doklady Akad. Nauk SSSR (N.S.) **63**, 479–482 (1948).
5. GERONIMUS, YA. L.: Sur quelques propriétés extrémales des polynomes trigonométriques. C. R. Acad. Sci. Paris **198**, 2221–2222 (1934).
6. AHIEZER, N. I., and M. G. KREĬN: Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremum-Problem. Zap. Har'kov. Mat. Obšč. **(4) 9**, 9–23 (1934) and **(4) 10**, 3–32 (1934).

3.5.37 Let the sequence $r_1, r_2, \ldots$ be monotonely decreasing and let $\lim_{n\to+\infty} r_n = 0$. Then, for $0 < \theta < 2\pi$,

$$\left|\sum_{k=n+1}^{+\infty} r_k e^{ik\theta}\right| \le \frac{r_n}{\sin\frac{\theta}{2}}. \tag{1}$$

Proof. If $0 < \theta < 2\pi$ and $m \ge 1$, then

$$\left|\sum_{k=n+1}^{n+m} e^{ik\theta}\right| = \left|\frac{e^{i(n+1)\theta}(1-e^{im\theta})}{1-e^{i\theta}}\right| \le \frac{2}{|1-e^{i\theta}|} = \frac{1}{\sin\frac{\theta}{2}}.$$

Therefore, by ABEL's inequality (see 2.2),

$$\left|\sum_{k=n+1}^{n+m} r_k e^{ik\theta}\right| \le \frac{r_{n+1}}{\sin\frac{\theta}{2}} < \frac{r_n}{\sin\frac{\theta}{2}}.$$

Letting $m \to +\infty$, we get (1).

Reference

BLANKFIELD, J., and D. ZEITLIN: Problem E 1220. Amer. Math. Monthly **64**, 47–48 (1957).

3.5.38 For all real x and $r = 1, 2, \ldots$,

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \left(\frac{\sin \pi n x}{n}\right)^{2r} \ge 0.$$

This inequality, due to J. N. LYNESS and C. MOLER, is generalized as follows:

If $0 < x_i < \pi$ and $N = 1, \ldots, k$, where $k = 1, 2, \ldots$, then

$$\sum_{n=1}^{N} \prod_{i=1}^{k} \frac{\sin n x_i}{n} > 0.$$

If $0 < x_i < \pi$ and $k = 3, 4, \ldots$, then

$$\sum_{n=1}^{+\infty} n \prod_{i=1}^{k} \frac{\sin n x_i}{n} > 0.$$

References

LYNESS, J. N., and C. MOLER: Problem 67-6. SIAM Review **9**, 250 (1967) and **11**, 82–86 (1969).

ASKEY, R., and J. FITCH: Some positive trigonometric sums. Notices Amer. Math. Soc. **15**, 769 (1968).

3.5.39 For $0 < x < \pi$, $-\pi < t < +\pi$, $|t| \neq x$ and $t \neq 0$ we have

$$\sum_{k=1}^{+\infty} \frac{\sin kx}{k} \frac{\sin\left(k - \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} > 0.$$

For $0 < x < \pi$, $0 < t < \pi$ and $t \neq x$, we have

$$\sum_{k=1}^{+\infty} \frac{\sin kx}{k} \frac{\cos \frac{t}{2} - \cos\left(k + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} > 0.$$

Reference

Hyltén-Cavallius, C.: A positive trigonometrical kernel. C. R. 12e Congrès Math. Scandinaves, Lund 1953, pp. 90–94 (1954).

3.5.40 Let the sequence $a = (a_0, a_1, \ldots)$ be four times monotone, i.e., let

$$\binom{k}{0} a_n - \binom{k}{1} a_{n+1} + \cdots + (-1)^k \binom{k}{k} a_{n+k} \geq 0 \quad (k = 1, 2, 3, 4).$$

Then, for $0 < x < \pi$,

$$\sum_{k=1}^{+\infty} a_k \sin kx \leq \frac{a_1}{2} \cot \frac{x}{2}. \tag{1}$$

Remark. Inequality (1) together with a number of other inequalities, was proved by M. Tomić in [1], who used geometrical reasoning. Almost at the same time, similar methods were used by C. Hyltén-Cavallius [2].

References

1. Tomić, M.: Sur les sommes trigonométriques à coefficients monotones (Serbian). Srpska Akad. Nauka. Zbornik Radova 18, Mat. Inst. 2, 13–52 (1952).
2. Hyltén-Cavallius, C.: Geometrical methods applied to trigonometrical sums. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar 21, No. 1, 19 pp. (1950).

3.5.41 If $T_{n-1}(x)$ is a trigonometric polynomial of order $n - 1$, then

$$\max_{-\pi \leq x \leq \pi} |T_{n-1}(x)| \leq n \max_{-\pi \leq x \leq \pi} |T_{n-1}(x) \sin x.|$$

Reference

Bari, N. K.: Generalization of inequalities of S. N. Bernštein and A. A. Markov (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 18, 159–176 (1954).

3.5.42 Let $T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$. Then, if $|T_n(x)| \leq 1$, we have

$$\frac{1}{2}|a_0| + \sum_{k=1}^{n} (|a_k| + |b_k|) \leq A n^{1/2},$$

where A is a constant.

Reference

Zygmund, A.: Trigonometric Series, vol. 1. Cambridge 1959, p. 244.

3.5.43 Let $S_n(x)$ be the n-th partial sum of FOURIER series of $f \in L^p$ $(p > 1)$. Then

$$\left(\int_{-\pi}^{+\pi} |S_n(x)|^p\, dx\right)^{1/p} \leq C_p \left(\int_{-\pi}^{+\pi} |f(x)|^p\, dx\right)^{1/p},$$

where C_p depends only on p.

Reference

ZYGMUND, A.: Trigonometric Series, vol. 1. Cambridge 1959, p. 266.

3.5.44 For any trigonometric polynomial $T_n(x)$,

$$\left(\int_{-\pi}^{+\pi} |T_n(x)|^q\, dx\right)^{1/q} \leq 2n^{1/p-1/q} \left(\int_{-\pi}^{+\pi} |T_n(x)|^p\, dx\right)^{1/p}$$

for $1 \leq p < q < +\infty$.

Reference

NIKOL'SKIĬ, S. M.: Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables (Russian). Trudy Mat. Inst. Steklov **38**, 244–278 (1951).

3.5.45 Let (a_k) be a sequence such that $a_k \geq 0$. If $a_k k^{-s}$ is decreasing for $s \geq 0$, (a_k) is said to belong to the class A_s. If $a_k k^s$ is increasing for some $s > 0$, (a_k) is said to belong to the class A_{-s}.

If $x \neq 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$, then

$$\left|\sum_{k=n}^{m} a_k f(kx)\right| \leq \begin{cases} \dfrac{a_n}{\left|\sin \dfrac{x}{2}\right|} \left(\dfrac{m}{n}\right)^s & \text{for} \quad (a_k) \in A_s, \\[2ex] \dfrac{a_m}{\left|\sin \dfrac{x}{2}\right|} \left(\dfrac{m}{n}\right)^s & \text{for} \quad (a_k) \in A_{-s}, \end{cases}$$

where $f(x) = \cos x$ or $f(x) = \sin x$.

Reference

LEBED', G. K.: On trigonometric series with coefficients which satisfy some conditions (Russian). Mat. Sb. **74 (116)**, 100–118 (1967).

3.5.46 Let $A_n^0 = 1$, $A_n^p = A_0^{p-1} + A_1^{p-1} + \cdots + A_n^{p-1}$ for $n = 0, 1, \ldots$ and $p = 1, 2, \ldots$ Then, if $t \in (0, \pi)$, $n = 0, 1, \ldots$ and if r is an integer ≥ 3, we have

$$\frac{1}{2}\cot\frac{t}{2} - \frac{1}{A_n^r} \sum_{\nu=0}^{n} A_{n-\nu}^{r-1} \sum_{\mu=0}^{\nu} \sin \mu t \geq 0.$$

Reference

TABERSKI, R.: Asymptotic formulae for Cesàro singular integrals. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10**, 637–640 (1962).

3.6 Inequalities Involving Exponential, Logarithmic and Gamma Functions

3.6.1 If $a > 0$ and $x > 0$, then

$$e^x \geq \left(\frac{ex}{a}\right)^a, \quad e^{-x} \leq \left(\frac{a}{ex}\right)^a,$$

$$\log x \leq \frac{a}{e} \sqrt[a]{x}, \quad -\log x \leq \frac{1}{eax^a}.$$

These inequalities are of importance for large values of a.

3.6.2 If a and t are real numbers such that $a \geq 1$ and $|t| \leq a$, then

$$0 \leq e^{-t} - \left(1 - \frac{t}{a}\right)^a \leq \frac{t^2 e^{-t}}{a}.$$

If $0 \leq t \leq a$ and $a \geq 2$, then

$$0 \leq e^{-t} - \left(1 - \frac{t}{a}\right)^a \leq \frac{t^2(1+t)\, e^{-t}}{2a}.$$

If $0 \leq t \leq a$ and $a > 0$, then

$$0 \leq e^{-t} - \left(1 - \frac{t}{a}\right)^a \leq \frac{t^2}{2a}.$$

Remark. These inequalities were the object of a sharp discussion between G. N. Watson and E. H. Neville.

References

Watson, G. N.: An inequality associated with gamma function. Messenger Math. **45**, 28–30 (1916).

Neville, E. H.: Two inequalities used in the theory of the gamma function. Math. Gaz. **20**, 279–280 (1936).

Neville, E. H.: Addition to Note 1209. Math. Gaz. **21**, 55–56 (1937).

Watson, G. N.: Comments on Note 1225. Math. Gaz. **21**, 292–295 (1937).

Dunkel, O.: Problem 3766 (Editorial note). Amer. Math. Monthly **45**, 58 (1938).

Whittaker, E. T., and G. N. Watson: A Course of Modern Analysis. Cambridge 1952, p. 242.

3.6.3 Let $S_n(x) = 1 + x + x^2/2! + \cdots + x^n/n!$. If n is a natural number and $x \geq 0$, then

(1) $$0 \leq e^x - \left(1 + \frac{x}{n}\right)^n \leq e^x \left[1 - \frac{1}{\left(1 + \frac{x}{n}\right)^{x/2}}\right],$$

(2) $$e^x - S_n(x) \leq \frac{e^x x}{n}.$$

Proof. Inequalities (1) are equivalent to

$$\left(1+\frac{x}{n}\right)^n \leq e^x \leq \left(1+\frac{x}{n}\right)^{n+x/2}.$$

We shall prove more general inequalities

$$\left(1+\frac{x}{p}\right)^p < e^x < \left(1+\frac{x}{p}\right)^{p+x/2} \quad (x>0 \quad \text{and} \quad p>0).$$

Let

$$F(p) = \left(1+\frac{x}{p}\right)^p, \quad G(p) = \left(1+\frac{x}{p}\right)^{p+x/2},$$

$$f(p) = \log F(p), \quad g(p) = \log G(p).$$

Since, for $p \to +\infty$,

$$\left(1+\frac{x}{p}\right)^p \to e^x \quad \text{and} \quad \left(1+\frac{x}{p}\right)^{x/2} \to 1,$$

it is sufficient to prove that F is an increasing and G a decreasing function of p, both for $p > 0$. This is equivalent to the assertion that f is an increasing and g a decreasing function, both for $p > 0$.

By differentiation we get

$$f'(p) = -\frac{x}{p+x} + \log\frac{p+x}{p}, \quad f''(p) = -\frac{x^2}{p(p+x)^2} < 0,$$

$$g'(p) = -\frac{x}{p}\,\frac{p+\frac{x}{2}}{p+x} + \log\frac{p+x}{2}, \quad g''(p) = \frac{x^3}{2p^2(p+x)^2} > 0.$$

It follows that

$$f'(p) > f'(+\infty) = 0, \quad g'(p) < g'(+\infty) = 0,$$

which proves the above result.

Inequality (2) is equivalent to

$$\sum_{k=n}^{+\infty} \frac{x^{k+1}}{(k+1)!} \leq \sum_{k=0}^{+\infty} \frac{x^{k+1}}{k!\,n},$$

i.e., to

$$\sum_{k=0}^{n-1} \frac{x^{k+1}}{k!\,n} + \sum_{k=n}^{+\infty} \frac{k+1-n}{(k+1)!\,n} x^{k+1} \geq 0,$$

which is obviously true for all $x \geq 0$.

Remark. Inequalities (1) and (2) are due to W. E. Sewell [1]. Inequalities (1) also contain the following (see [2]):

$$(3) \qquad 0 \leq e^x - \left(1+\frac{x}{n}\right)^n \leq \frac{e^x x^2}{n}, \quad n \geq 1, \quad 0 \leq x \leq n.$$

Combining inequalities (1) and (2) we get

$$\left|\left(1+\frac{x}{n}\right)^n - S_n(x)\right| \le e^x\left[1+\frac{x}{n}-\frac{1}{\left(1+\frac{x}{n}\right)^{x/2}}\right], \qquad x \ge 0,$$

while (2) and (3) together yield

$$\left|\left(1+\frac{x}{n}\right)^n - S_n(x)\right| \le \frac{e^x x}{n}(1+x), \quad 0 \le x \le n.$$

References

1. Sewell, W. E.: Some inequalities connected with exponential function (Spanish). Rev. Ci. (Lima) **40**, No. 425, 453–456 (1938).
2. Dunkel, O.: Problem 3766 (Editorial note). Amer. Math. Monthly **45**, 58 (1938).

3.6.4 If $x > 0$, then

$$e^x > 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^n}{n!}\left(1+\frac{x^2}{(n+1)^2}\right)^{1/2}.$$

This inequality is a consequence of

$$2I_{n+1}I_{n-1} > I_n^2,$$

where

$$I_n = e^x - \left(1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!}\right) \quad \text{for} \quad x > 0.$$

Reference

Kesava Menon, P.: Some integral inequalities. Math. Student **11**, 36–38 (1943).

3.6.5 For all real x, the following inequality holds:

$$(1) \qquad |e^x(12-6x+x^2)-(12+6x+x^2)| \le \frac{1}{60}|x|^5 e^{|x|}.$$

Proof. Consider the following identity

$$\frac{e^x - (1+x+x^2/2)}{x^3} = \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \cdots.$$

Differentiating twice the above identity, we get

$$\frac{e^x(12-6x+x^2)-(12+6x+x^2)}{x^5} = \frac{2}{5!} + \cdots + \frac{(n+1)(n+2)}{(n+5)!}x^n + \cdots$$

$$= \frac{1}{60} + \cdots + \frac{1}{(n+3)(n+4)(n+5)}\frac{x^n}{n!} + \cdots$$

$$= \frac{1}{60}\left(1 + \cdots + \frac{60}{(n+3)(n+4)(n+5)}\frac{x^n}{n!} + \cdots\right).$$

However, since $\frac{60}{(n+3)(n+4)(n+5)} \frac{|x|^n}{n!} \leq \frac{|x|^n}{n!}$, we obtain

$$60 \left| \frac{e^x(12 - 6x + x^2) - (12 + 6x + x^2)}{x^5} \right| \leq 1 + \cdots + \frac{|x|^n}{n!} + \cdots = e^{|x|},$$

which yields inequality (1).

The above proof of inequality (1) is due to F. Motte: Problem 3707. Revue Math. Spéc. **46**, 129 (1935/36).

G. Kalajdžić gave the following generalization of (1):

For any real x and any natural number n, we have

$$\left| \sum_{k=0}^{n} \left((-1)^k e^x + (-1)^{n+1}\right) \binom{n+k}{n} \frac{x^{n-k}}{(n-k)!} \right| \leq \frac{1}{(2n+1)!} |x|^{2n+1} e^{|x|}.$$

The proof is similar to the one given above.

3.6.6 For the exponential function the following rational approximations are valid:

$$\frac{P_{2k}(x)}{P_{2k}(-x)} \leq e^x \leq \frac{P_{2k+1}(x)}{P_{2k+1}(-x)} \qquad (0 \leq x < x_n),$$

where

$$P_n(x) = \frac{1}{n!} \sum_{\nu=0}^{n} (2n - \nu)! \binom{n}{\nu} x^\nu,$$

x_n being the least positive zero of $P_n(-x)$;

$$e^x \leq \frac{2+x}{2-x} \qquad (0 \leq x < 2);$$

$$1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2} + \frac{x^3}{2(3-x)} \qquad (0 \leq x < 3);$$

$$e^x \leq \frac{n}{n-x} \cdot \frac{x^n}{n!} + \sum_{\nu=0}^{n-1} \frac{x^\nu}{\nu!} \qquad (0 \leq x < n).$$

Reference

Karamata, J.: Sur l'approximation de e^x par des fonctions rationnelles (Serbian). Bull. Soc. Math. Phys. Serbie **1**, No. **1**, 7–19 (1949).

3.6.7 If n is a natural number, then

$$\sum_{r=0}^{n} \frac{x^r}{r!} < e^x \quad \text{for all} \quad x > 0, \quad \text{and} \quad e^x < \sum_{r=0}^{n} \frac{x^r}{r!} + \frac{x^{n+1}}{(n-x+1)\,(n!)}$$

for $0 < x < n + 1$.

Proof. Setting

$$f(x) = e^{-x} \sum_{r=0}^{n} \frac{x^r}{r!}, \; g(x) = f(x) + e^{-x} \frac{x^{n+1}}{(n-x+1)\,(n!)},$$

we have $f'(x) < 0$ for all $x > 0$ and $g'(x) > 0$ for $0 < x < n + 1$.

Hence, we obtain the desired result.

Remark. See the last inequality in 3.6.6.

3.6.8 If $a, b \geq 0$, then

$$|\cosh a - \cosh b| \geq |a - b| \sqrt{\sinh a \sinh b}.$$

Reference

VAN DER CORPUT, J. G.: Problem 96. Wisk. Opgaven **16**, 243—244 (1935).

3.6.9 If $x \neq 0$, then

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3. \tag{1}$$

The exponent 3 is the least possible.

Proof. Let us put $f(x) = x - \sinh x (\cosh x)^{-1/3}$. We find that $f(0) = f'(0) = 0$, and

$$f''(x) = -\frac{4}{9} \sinh^3 x (\cosh x)^{-7/3}.$$

Hence $f(x)$ is concave on $(0, +\infty)$ and its graph is tangent to the x-axis at the origin. Therefore $f(x) < 0$ on $(0, +\infty)$ which implies that (1) holds for $0 < x < +\infty$. All that remains is to note that both members of (1) are even functions.

Since

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots, \quad \text{and} \quad \left(\frac{\sinh x}{x}\right)^a = 1 + \frac{a}{3}\frac{x^2}{2} + \cdots,$$

for x in the neighbourhood of the origin, we infer that $a = 3$ is the smallest value of a such that

$$\cosh x < \left(\frac{\sinh x}{x}\right)^a \quad (x \neq 0).$$

Inequality (1) and its proof are due to I. LAZAREVIĆ.

3.6.10 If $x \geq 0$, then

$$\arctan x \leq \frac{\pi}{2} \tanh x.$$

Reference

GARNIR, H. G.: Fonctions de variables réelles, vol. 1. Louvain-Paris 1963, p. 268.

3.6.11 For $x > 0$ we have

$$\tanh x > \sin x \cos x.$$

References

LOCHS, G.: Die Konvergenzradien einiger zur Lösung transzendenter Gleichungen verwendeter Potenzreihen. Monatsh. Math. **58**, 118—122 (1954).

LAZAREVIĆ, I.: Sur une inégalité de Lochs. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **230—241**, 55—56 (1968).

3.6.12 If $0 < x < 5$, then

$$\frac{(3 + x^2/11)\sinh x}{2 + \cosh x + x^2/11} < x < \frac{(3 + x^2/10)\sinh x}{2 + \cosh x + x^2/10}.$$

Reference

Frame, J. S.: Some trigonometric, hyperbolic and elliptic approximations. Amer. Math. Monthly **61**, 623–626 (1954).

3.6.13 If $x > 0$ and $y > 0$, then

(1) $$\frac{1 - e^{-x-y}}{(x + y)(1 - e^{-x})(1 - e^{-y})} - \frac{1}{xy} \leq \frac{1}{12}.$$

Proof. This inequality is equivalent to

$$\frac{1}{1 - e^{-x}} + \frac{1}{1 - e^{-y}} - 1 \leq \frac{1}{x} + \frac{1}{y} + \frac{x + y}{12},$$

i.e.,

$$f(x) + f(y) \leq 1,$$

where

$$f(x) = \frac{1}{1 - e^{-x}} - \frac{1}{x} - \frac{x}{12}.$$

Since $f(x) \leq 1/2$, inequality (1) is proved.

References

Hardy, G. H.: A note on two inequalities. J. London Math. Soc. **11**, 167–170 (1936).
Bromwich, T. J. I'a.: Infinite Series. 2nd ed., London 1926, p. 299.

3.6.14 For $x > 0$ and $r > 1$, we have

(1) $$\sinh^{r+1} rx + \cosh^{r+1} rx \leq \cosh^r (r + 1)\, x,$$

while for $x > 0$ and $0 < r < 1$,

(2) $$\sinh^{r+1} rx + \cosh^{r+1} rx \geq \cosh^r (r + 1)\, x.$$

Equality holds either if $r = 0$ or $r = 1$, or if $x = 0$.
The substitution $y = e^{-2x}$ reduces inequality (1) to the algebraic form

(3) $$(1 - y^r)^{r+1} + (1 + y^r)^{r+1} \leq 2(1 + y^{r+1})^r,$$

valid for $0 < y < 1$ and $r > 1$. The inequality is reversed for $0 < r < 1$ and the equality holds on all the boundaries.

Reference

Frame, J. S.: Problem 3764. Amer. Math. Monthly **45**, 54–56 (1938).

3.6.15 If $x > 0$ and $x \neq 1$, then

$$\frac{\log x}{x-1} \le \frac{1}{\sqrt{x}}. \tag{1}$$

Proof. Putting $x = (1+t)^2/(1-t)^2$, inequality (1) becomes

$$\frac{1}{t}\log\frac{1+t}{1-t} - \frac{2}{1-t^2} \le 0 \quad \text{for} \quad 0 < |t| < 1.$$

Using the power series expansions, we get

$$\sum_{n=1}^{+\infty}\left(1 - \frac{1}{2n+1}\right)t^{2n} \ge 0 \quad \text{for} \quad |t| < 1,$$

which is evidently true.

Inequality (1) is due to J. Karamata (see 3.6.16), and the above proof to B. Mesihović.

3.6.16 If $x > 0$, and $x \neq 1$, then

$$\frac{\log x}{x-1} \le \frac{1+x^{1/3}}{x+x^{1/3}}. \tag{1}$$

Proof. Setting $x = (1+t)^3/(1-t)^3$, inequality (1) becomes

$$\frac{3}{2t}\log\frac{1+t}{1-t} - \frac{t^2+3}{1-t^4} \le 0 \quad \text{for} \quad 0 < |t| < 1.$$

Using power series expansions, we get

$$\sum_{n=0}^{+\infty} t^{4n+2} + 3\sum_{n=0}^{+\infty} t^{4n} - 3\sum_{n=0}^{+\infty}\frac{t^{2n}}{2n+1} \ge 0 \quad \text{for} \quad |t| < 1,$$

i.e.,

$$3\sum_{n=0}^{+\infty}\left(1 - \frac{1}{4n+1}\right)t^{4n} + \sum_{n=1}^{+\infty}\left(1 - \frac{3}{4n+3}\right)t^{4n+2} \ge 0,$$

which is obviously true.

Inequality (1) is due to J. Karamata, and the above proof to B. Mesihović.

Remark. Starting with (1), which is very precise because

$$\frac{\log x}{x-1} - \frac{1+x^{1/3}}{x+x^{1/3}} \sim \frac{(x-1)^4}{1620} \quad \text{for} \quad x \to 1,$$

J. Karamata demonstrated that $1/3 < f(n) < 1/2$, where

$$f(n) = 1 + \frac{1}{2}\frac{e^n n!}{n^n} - \frac{n!}{n^n}\left(1 + \frac{n}{1!} + \cdots + \frac{n^n}{n!}\right) \quad \text{with} \quad n = 1, 2, \ldots$$

J. Karamata proved, in fact, that f is a monotonely decreasing function from $1/2$ to $1/3$, when n increases from 0 to $+\infty$.

Compare with 3.9.7.

Reference

KARAMATA, J.: Sur quelques problèmes posés par Ramanujan. J. Indian Math. Soc. (N.S.) **24**, 343–365 (1960).

3.6.17 If $0 < y < x$, then

$$\frac{x+y}{2} > \frac{x-y}{\log x - \log y}. \tag{1}$$

Proof. We integrate the inequality

$$\frac{1}{2t} > \frac{2}{(1+t)^2} \qquad (t > 1),$$

and we get

$$\frac{1}{2}\int_1^{x/y} \frac{dt}{t} > 2\int_1^{x/y} \frac{1}{(1+t)^2}\,dt,$$

i.e.,

$$\frac{1}{2}\log\frac{x}{y} > \frac{x-y}{x+y} \quad \text{for} \quad 0 < y < x.$$

This inequality is equivalent to (1).

Remark. This proof is due to B. MESIHOVIĆ. Two other proofs are given in MITRINOVIĆ 1, p. 158, and in MITRINOVIĆ 2, pp. 192–193.

3.6.18 For $x > 0$,

$$\frac{2}{2x+1} < \log\left(1+\frac{1}{x}\right) < \frac{1}{\sqrt{x^2+x}}. \tag{1}$$

Proof. Consider first the function f defined by

$$f(x) = \frac{2}{2x+1} - \log\left(1+\frac{1}{x}\right) \quad \text{for} \quad x > 0.$$

Since

$$f'(x) = \frac{1}{x(1+x)\,(2x+1)^2} > 0 \quad \text{for} \quad x > 0,$$

f is an increasing function. From this fact and since $\lim_{x\to+\infty} f(x) = 0$, we conclude that $f(x) < 0$ for $x > 0$.

Now, consider the function g defined by

$$g(x) = \log\left(1+\frac{1}{x}\right) - \frac{1}{\sqrt{x+x^2}} \quad \text{for} \quad x > 0.$$

Since

$$g'(x) = \frac{2x+1}{2(x^2+x)^{3/2}} - \frac{1}{x(1+x)} > 0 \quad \text{for} \quad x > 0,$$

we see that g is an increasing function. By this fact and since $\lim_{x\to+\infty} g(x) = 0$, we obtain that $g(x) < 0$ for $x > 0$.

This establishes (1).

Comment by P. M. VASIĆ. If in the second inequality of (1), for $x > 1$, we replace x by $1/(x - 1)$, and for $0 < x < 1$ we replace x by $x/(1 - x)$, we obtain

$$\frac{\log x}{x-1} < \frac{1}{\sqrt{x}} \qquad \text{(see 3.6.15)}.$$

Replacing $1 + (1/x)$ by x/y ($x > y > 0$) in the first inequality of (1), we have

$$\frac{x+y}{2} > \frac{x-y}{\log x - \log y} \qquad \text{(see 3.6.17)}.$$

3.6.19 For $x > 0$,

$$(1) \qquad \frac{2}{2x+1} < \log\left(1 + \frac{1}{x}\right) < \frac{2}{2x+1}\left(1 + \frac{1}{12x} - \frac{1}{12(x+1)}\right).$$

According to a remark by P. R. BEESACK, (1) can be improved to

$$\frac{2}{2x+1}\left(1 + \frac{1}{12x} - \frac{1}{12(x+1)} - \frac{1}{360x^3} + \frac{1}{360(x+1)^3}\right) < \log\left(1 + \frac{1}{x}\right)$$
$$< \frac{2}{2x+1}\left(1 + \frac{1}{12x} - \frac{1}{12(x+1)} - \frac{1}{(360+\gamma(x))\,x^3} + \frac{1}{(360+\gamma(x))\,(x+1)^3}\right),$$

where

$$\gamma(x) = 30\,\frac{7x^2 + 7x + 1}{x^2(x+1)^2},$$

so that for large x the upper and lower bound are very close. These results can be used to give a good form of STIRLING's formula.

Remark 1. Inequalities (1) play an important role in the theory of gamma function.

Remark 2. The second inequality in (1) cannot be compared with the second inequality in 3.6.18.

Reference

CAMPBELL, R.: Les intégrales Eulériennes et leurs applications. Paris 1966.

3.6.20 If $a > 1$, $p > 0$ and $0 < x < 1$, then

$$0 < \log_a(p + x) - \log_a p - x[\log_a(p+1) - \log_a p]$$
$$< \frac{x(1-x)}{2p^2}\log_a e \leq \frac{\log_a e}{8p^2}.$$

Reference

LEFORT, G.: Algèbre et analyse: exercices. Paris 1964, p. 229.

3.6.21 If $x + b > 1$, $x + a > 0$ and $x < y$, then

$$\log_{x+b}(x+a) > \log_{y+b}(y+a) \qquad (a > b),$$
$$\log_{x+b}(x+a) < \log_{y+b}(y+a) \qquad (a < b).$$

In particular,

$$\log_x(x+1) > \log_y(y+1) \qquad (1 < x < y).$$

3.6.22 If $n \geq 2$ is a positive integer and $x > 0$, then

(1)
$$(x+n-1)\log(x+n-1) - x\log x$$
$$< n - 1 + \log[(x+1)(x+2)\cdots(x+n-1)]$$
$$< (x+n)\log(x+n) - (x+1)\log(x+1).$$

Proof. Let us put
$$f(x) = (x+n-1)\log(x+n-1) - x\log x - \log[(x+1)(x+2)\cdots(x+n-1)]$$
and
$$g(x) = (x+n)\log(x+n) - (x+1)\log(x+1) - \log[(x+1)(x+2)\cdots(x+n-1)].$$

Differentiating with respect to x, we get
$$f'(x) = \log(x+n-1) - \log x - \frac{1}{x+1} - \frac{1}{x+2} - \cdots - \frac{1}{x+n-1}$$
$$= \sum_{k=0}^{n-2}\left[\log(x+k+1) - \log(x+k) - \frac{1}{x+k+1}\right]$$
and
$$g'(x) = \log(x+n) - \log(x+1) - \frac{1}{x+1} - \frac{1}{x+2} - \cdots - \frac{1}{x+n-1}$$
$$= \sum_{k=1}^{n-1}\left[\log(x+k+1) - \log(x+k) - \frac{1}{x+k}\right].$$

By the LAGRANGE mean value theorem we have
$$\log(x+k+1) - \log(x+k) = \frac{1}{x+k+\theta} \qquad (0 < \theta < 1),$$
which implies that
$$f'(x) > 0 \quad \text{and} \quad g'(x) < 0 \quad (0 < x < +\infty).$$

Therefore, we have

(2) $$f(x) < f(+\infty) \quad \text{and} \quad g(x) > g(+\infty) \quad (0 < x < +\infty).$$

We compute
$$f(+\infty) = \lim_{x\to+\infty}\left[x\log\frac{x+n-1}{x} - \log\frac{(x+1)(x+2)\cdots(x+n-1)}{(x+n-1)^{n-1}}\right]$$
$$= n - 1,$$
since
$$\lim_{x\to+\infty} x\log\frac{x+n-1}{x} = n-1$$
and
$$\lim_{x\to+\infty}\frac{(x+1)(x+2)\cdots(x+n-1)}{(x+n-1)^{n-1}} = 1.$$

In the same way we find that $g(+\infty) = n - 1$. Substituting these values into (2), we obtain

$$f(x) < n - 1 < g(x) \quad (0 < x < +\infty)$$

which is equivalent to (1).

3.6.23 If $x > 1$, then

$$(x-1)\sum_{k=0}^{n-1} \frac{1}{n + (k+1)(x-1)} < \log x < (x-1)\sum_{k=0}^{n-1} \frac{1}{n + k(x-1)}.$$

3.6.24 If $0 < a < 1$ and $x > 0$ with $x \neq 1$, then

$$(x+1)^{a-1} > \frac{x^a - 1}{x - 1}. \tag{1}$$

Proof. It is sufficient to prove (1) for $x \in (0, 1)$ since (1) is invariant under the substitution $x \to 1/x$.

If $x \in (0, 1)$, (1) is equivalent to

$$(1+x)^{1-a}(1-x^a) < 1 - x. \tag{2}$$

Let $f(x) = (1+x)^{1-a}(1-x^a)$; then we compute that

$$f''(x) = a(1-a)\,x^{a-2}(1+x)^{-1-a}(1-x^{2-a}).$$

Hence f is strictly convex on $[0, 1]$. Since $f(0) = 1$ and $f(1) = 0$, we infer that (2) is true.

Reference

Kober, H.: Approximation by integral functions in the complex domain. Trans. Amer. Math. Soc. **56**, 22 (1944).

3.6.25 If $p > 1$ and $0 \le x \le 1$, then

$$2^{-(p-1)} \le x^p + (1-x)^p \le 1.$$

3.6.26 If $a \ge 1$, $p \ge q > 0$ and $p - q \le 1$, then

$$a^{p+q} - 1 \ge (p+q)(a^p - a^q). \tag{1}$$

In particular, for $p = (n+1)/2$ and $q = (n-1)/2$ (n is a natural number), inequality (1) becomes

$$a^n - 1 \ge n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right) \quad \text{for} \quad a \ge 1.$$

3.6.27 Let $x \ge 0$ and $y \ge 0$ and if $0 < r < 1$, then

$$(1+x)^r (1+y)^{1-r} \ge 1 + x^r y^{1-r},$$

with equality if and only if $x = y$.

3.6.28 If a, b, x are real numbers such that $0 < a < b$ and $0 < x < 1$, then

(1) $$\left(\frac{1 - x^b}{1 - x^{a+b}}\right)^b > \left(\frac{1 - x^a}{1 - x^{a+b}}\right)^a.$$

Proof. We can write (1) in the form

(2) $$b \log(1 - x^b) - a \log(1 - x^a) - (b - a) \log(1 - x^{a+b}) > 0.$$

Using series representation of $\log(1 + x)$, inequality (2) becomes

$$\sum_{k=1}^{+\infty} \frac{(b - a)\, x^{(a+b)k} + a x^{ak} - b x^{bk}}{k} > 0.$$

Hence, it is sufficient to prove that

$$(b - a)\, t^{b+a} + a t^a - b t^b > 0 \qquad (0 < t < 1),$$

i.e.,

$$\frac{1 - t^b}{1 - t^a} > \frac{b}{a}\, t^{b-a} \qquad (0 < t < 1).$$

The last inequality is a simple consequence of CAUCHY's theorem of differential calculus:

$$\frac{1 - t^b}{1 - t^a} = \frac{b\theta^{b-1}}{a\theta^{a-1}} = \frac{b}{a}\, \theta^{b-a} > \frac{b}{a}\, t^{b-a} \qquad (t < \theta < 1).$$

Hence, the inequality (1) is established.

The above result is due to D. S. MITRINOVIĆ and D. Ž. DJOKOVIĆ.

Remark. Inspired by the method of proving the above inequality, as well as 3.6.15 and 3.6.16, S. B. PREŠIĆ has communicated to us the following result.

Let $x > 0$ and let $\alpha_1 > \cdots > \alpha_n > 0$. Let

$$P(x) = A_1 x^{\alpha_1} + \cdots + A_n x^{\alpha_n}$$

and let (C) be a condition for A_i and α_j such that

(1) $$P(x) \geq 0 \quad \text{for all} \quad 0 < x < 1,$$

whenever (C) is fulfilled. Then the following inequality holds

(2) $$A_1 f(x^{\alpha_1}) + \cdots + A_n f(x^{\alpha_n}) \geq 0 \quad (0 < x < 1),$$

where f is an arbitrary function which can in $(0, 1)$ be written in the form

(3) $$f(x) = \sum_{n=0}^{+\infty} a_n x^n \text{ with } a_n \geq 0.$$

As a special case, if $f(x) = -\log(1 - x)$ according to (2) we get

(4) $$(1 - x^{\alpha_1})^{A_1} \cdots (1 - x^{\alpha_n})^{A_n} \leq 1 \quad (0 \leq x < 1).$$

Examples of use. 1° Let

$$P(x) = (b - a)\,x^{b+a} + ax^a - bx^b,$$

where x, a, b are real positive numbers. Then, as can easily be checked, a condition (C) is given by: $b > a > 0$. From inequality (4) we get

$$\left(\frac{1 - x^b}{1 - x^{a+b}}\right)^b > \left(\frac{1 - x^a}{1 - x^{a+b}}\right)^a, \tag{5}$$

where $0 < a < b$ and $0 < x < 1$.

Here the sign $>$ was written instead of $\geq$, since in this case the same replacement of signs can be done in (1).

2° Let $P(x) = (x^2 + 2px + q)\,x$. A condition (C) is

$$(q \geq 0 \wedge 1 + 2p + q \geq 0) \wedge (-1 \leq p \leq 0 \Rightarrow p^2 \leq q).$$

According to (2), if the condition (C) is fulfilled, we have

$$f(x^3) + 2pf(x^2) + qf(x) \geq 0 \quad (0 < x < 1),$$

where f is an arbitrary function satisfying (3).

3.6.29 Let m, n, p, q be positive numbers with $m < n$. If $p > q$, then

$$(1 - x^p)^m > (1 - x^q)^n \qquad (0 < x < 1). \tag{1}$$

If $p < q$, then

$$(1 - x^p)^m < (1 - x^q)^n \qquad (0 < x < x_0), \tag{2}$$

$$(1 - x^p)^m > (1 - x^q)^n \qquad (x_0 < x < 1), \tag{3}$$

where $x_0 \in (0, 1)$ is the unique solution of

$$(1 - x^p)^m = (1 - x^q)^n.$$

Proof. If $p > q$, then $1 - x^p > 1 - x^q$ for $0 < x < 1$ and (1) is evidently true.

Putting $x^p = t$, $a = q/p$, $b = n/m$, we transform (2) and (3) into

$$1 - t < (1 - t^a)^b \qquad (0 < t < t_0), \tag{2'}$$

$$1 - t > (1 - t^a)^b \qquad (t_0 < t < 1), \tag{3'}$$

where $t_0 \in (0, 1)$ is the unique solution of

$$1 - t = (1 - t^a)^b.$$

For $p < q$ and $m < n$ we have $a > 1$ and $b > 1$. Let us consider

$$f(t) = (1 - t^a)^b \qquad (0 < t < 1)$$

and its derivatives

$$f'(t) = -ab(1 - t^a)^{b-1}\,t^{a-1},$$

$$f''(t) = ab(1 - t^a)^{b-2}\,t^{a-2}\big(t^a(ab - 1) - (a - 1)\big).$$

Hence, $f(t)$ is concave on $(0, t_1)$ and convex on $(t_1, 1)$, where

$$t_1 = \left(\frac{a-1}{ab-1}\right)^{1/a}.$$

From $f(0) = 1$, $f(1) = 0$, $f'(0) = f'(1) = 0$ we conclude that $f(\varepsilon) > 1 - \varepsilon$ and $f(1-\varepsilon) < \varepsilon$ for small positive values of ε. Under these conditions the graph of $f(t)$ intersects the graph of $1 - t$ in only one point $t_0 \in (0, 1)$ and

$$f(t) > 1 - t \quad \text{for} \quad 0 < t < t_0,$$
$$f(t) < 1 - t \quad \text{for} \quad t_0 < t < 1.$$

These inequalities are identical to (2′) and (3′).

This proof was communicated to us by E. K. Godunova. Inequalities (1), (2) and (3) are due to D. S. Mitrinović and D. Ž. Djoković.

See also Chr. Karanicoloff: Sur une inégalité concernant la fonction puissance. Mat. Vesnik **1 (16)**, 159–161 (1964).

3.6.30 If $0 < a < 1$, then

$$(1+a)^{1-a}(1-a)^{1+a} < 1 < (1+a)^{1+a}(1-a)^{1-a}.$$

3.6.31 If $a > 0$ and $x > 1$, then

(1) $$\frac{x^{a+1} - x^{-a-1}}{x^a - x^{-a}} > \frac{a+1}{ax}.$$

Proof. Simplifying (1), we get

$$ax^{a+2} + x^{-a} - (a+1)\,x^a > 0,$$

i.e.,

(2) $$\frac{a}{a+1}x^{a+2} + \frac{1}{a+1}x^{-a} > x^a.$$

Inequality (2) is a consequence of the arithmetic-geometric mean inequality. Indeed, the left-hand side of (2) is the arithmetic mean of x^{a+2} and x^{-a} with weights $a/(a+1)$ and $1/(a+1)$, and the right-hand side is its geometric mean with the same weights.

3.6.32 If $x > 0$, then

$$x^x \geq e^{x-1}.$$

3.6.33 Let $0 < x < e$; then

$$(e+x)^{e-x} > (e-x)^{e+x}.$$

3.6.34 If a, b and x are positive numbers, with $a \neq b$, then

$$\left(\frac{a+x}{b+x}\right)^{b+x} > \left(\frac{a}{b}\right)^b.$$

3.6.35 If $0 < a < 1$, then

$$\frac{2}{e} < a^{\frac{a}{1-a}} + a^{\frac{1}{1-a}} < 1. \tag{1}$$

Proof. Denoting

$$f(a) = (1 + a)\, a^{\frac{a}{1-a}} \qquad (0 < a < 1),$$

we obtain

$$\log f(a) = \log(1 + a) + \frac{a}{1 - a} \log a,$$

$$\frac{f'(a)}{f(a)} = \frac{1}{(1 - a)^2}\left(\log a + 2\frac{1 - a}{1 + a}\right) \equiv \frac{g(a)}{(1 - a)^2},$$

and

$$g'(a) = \frac{1}{a}\left(\frac{1 - a}{1 + a}\right)^2.$$

We infer that

$$g(a) < g(1) = 0, \qquad f'(a) < 0,$$

$$\lim_{a \to 1-} f(a) < f(a) < \lim_{a \to 0+} f(a).$$

The last inequality is identical to (1) since

$$\lim_{a \to 1-} f(a) = \frac{2}{e}, \qquad \lim_{a \to 0+} f(a) = 1.$$

The above result is due to D. S. MITRINOVIĆ and D. Ž. DJOKOVIĆ.

3.6.36 If $p + q = 1$, $0 < p < 1$, and if r, s are positive numbers greater than 1, then

$$(1 - p^r)^s + (1 - q^s)^r > 1.$$

Reference

TURNER, J. C., and V. CONWAY: Problem 68-1. SIAM Review **10**, 107–108 (1968).

3.6.37 If n is a positive integer and

$$f(a) = \frac{a(1 + 2^{a-1} + 3^{a-1} + \cdots + n^{a-1})}{n(n + 1)^{a-1}},$$

then

$$f(a) < 1 \quad \text{if} \quad 0 < a < 1 \quad \text{or} \quad a > 2,$$

$$f(a) > 1 \quad \text{if} \quad 1 < a < 2,$$

$$f(a) = 1 \quad \text{if} \quad a = 1 \quad \text{or} \quad a = 2.$$

Remark. These inequalities are due to C. RYLL-NARDZEWSKI. See the paper: M. BIERNACKI: Sur une inégalité de M. Riesz et sur quelques inégalités analogues Ann. Univ. Mariae Curie-Skłodowska **A 6**, 79–89 (1952).

3.6.38 If $a > 0$ and $b > 0$, then

$$(1) \qquad a^b + b^a > 1.$$

Proof. Inequality (1) holds if $a \geq 1$ or $b \geq 1$. We have to prove that it also holds if $0 < a < 1$ and $0 < b < 1$. We set

$$f(b) = a^b + b^a - 1 \qquad (0 < a < 1, \quad 0 < b < 1),$$

and obtain

$$f(0) = 0, \quad f(1) = a > 0, \quad f'(b) = a^b \log a + ab^{a-1}.$$

Assuming that (1) is false we infer that there exists $b \in (0, 1)$ such that $f'(b) = 0$ and $f(b) \leq 0$; i.e.,

$$(2) \qquad a^b \log a + ab^{a-1} = 0, \quad a^b + b^a - 1 \leq 0 \qquad (0 < a < 1).$$

From (2) we find

$$1 - b\frac{\log a}{a} - a^{-b} \leq 0 \qquad (0 < a < 1).$$

We shall prove that the last inequality cannot hold for any $a \in (0, 1)$ and $b \in (0, 1)$. If

$$g(b) = 1 - b\frac{\log a}{a} - a^{-b},$$

then

$$g'(b) = \log a\left(a^{-b} - \frac{1}{a}\right), \qquad g''(b) = -a^{-b}(\log a)^2.$$

Since $0 < a < 1$, we conclude that

$$g''(b) < 0, \quad g'(b) > g'(1) = 0, \quad g(b) > g(0) = 0.$$

Hence, this contradiction proves that (1) is true.

Remark. The above proof was communicated to us by R. Lučić. Three other proofs are published in Mitrinović 2, pp. 182–186.

3.6.39 1° If $p \geq 1$ is a real number and $a_k > 0$, $b_k > 0$ $(k = 1, \ldots, n)$, then

$$(1) \qquad \left|\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} - \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right| \leq \min\left\{\sum_{k=1}^{n} |a_k - b_{t(k)}| : t \in P\right\},$$

where P denotes the set of all permutations of the set $\{1, \ldots, n\}$.

2° Excluding the trivial cases $p = 1$ and $n = 1$, inequality (1) becomes an equality if and only if its right-hand side is equal to zero.

3° With the additional conditions,

$$\sum_{k=1}^{+\infty} a_k^p < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} b_k^p < +\infty,$$

statements 1° and 2° are true also for $n = +\infty$.

This result presents an answer of D. D. ADAMOVIĆ to Problem 109 of D. S. MITRINOVIĆ proposed in Mat. Vesnik **4 (19)**, 453 (1967), and solved in **5 (20)**, 548–549 (1968).

3.6.40 If $\alpha \geq 1$, $\alpha + \beta \geq 1$, and $x_k > 0$ for $k = 1, \ldots, n$, then

$$\sum_{i=1}^{n} x_i^{\alpha} \left(\sum_{j=1}^{i} x_j \right)^{\beta} \leq \left(\sum_{i=1}^{n} x_i \right)^{\alpha+\beta}.$$

Remark. This is one of the results obtained by P. R. BEESACK who studied inequalities of the form

$$\sum_{i=1}^{n} x_i^{\alpha} \left(\sum_{j=1}^{i} x_j \right)^{\beta} \leq A_n(\alpha, \beta) \left(\sum_{i=1}^{n} x_i \right)^{\alpha+\beta},$$

and

$$\sum_{i=1}^{n} x_i^{\alpha} \left(\sum_{j=1}^{i} x_j \right)^{\beta} \geq a_n(\alpha, \beta) \left(\sum_{i=1}^{n} x_i \right)^{\alpha+\beta},$$

where α and β are real parameters and $x_k > 0$ for $k = 1, \ldots, n$.

In view of the length of the discussion, we are not able to include all those interesting results.

Reference

BEESACK, P. R.: On certain discrete inequalities involving partial sums. Canad. J. Math. 21, 222–234 (1969).

3.6.41 If $x_k > 0$ $(k = 1, \ldots, n)$, $x_1 + \cdots + x_n = 1$ and $a > 0$, then

$$\sum_{k=1}^{n} \left(x_k + \frac{1}{x_k} \right)^{a} \geq \frac{(n^2+1)^a}{n^{a-1}}. \tag{1}$$

Proof. We shall apply the method of LAGRANGE multipliers. Let

$$F(x_1, \ldots, x_n; \lambda) = \sum_{k=1}^{n} \left(x_k + \frac{1}{x_k} \right)^{a} + \lambda \sum_{k=1}^{n} x_k.$$

The stationary points are determined by the following system of equations

$$\frac{\partial F}{\partial x_k} = a \left(x_k + \frac{1}{x_k} \right)^{a-1} \left(1 - \frac{1}{x_k^2} \right) + \lambda = 0 \qquad (k = 1, \ldots, n), \tag{2}$$

$$\sum_{k=1}^{n} x_k = 1. \tag{3}$$

First, we shall prove that the function

$$f(x) = \left(x + \frac{1}{x} \right)^{a-1} \left(\frac{1}{x^2} - 1 \right) \qquad (0 < x \leq 1)$$

is strictly decreasing. We have

$$f'(x) = \left(x + \frac{1}{x} \right)^{a-2} g(x) \tag{4}$$

with

(5) $$g(x) = (1-a)\left(\frac{1}{x^2} - 1\right)^2 - \frac{2}{x^3}\left(x + \frac{1}{x}\right),$$

i.e.,

(6) $$g(x) = 1 - a - \frac{1+a}{x^4} - 2\frac{2-a}{x^2}.$$

If $a \geq 1$ it is evident from (5) that $g(x) < 0$ for $0 < x < 1$. If $a < 1$ it is evident from (6) that g is increasing for $0 < x < +\infty$ which implies that

$$g(x) \leq g(1) = -4 \qquad (0 < x < 1).$$

In both cases $g(x) < 0$ in $(0, 1)$. So equality (4) gives $f'(x) < 0$ for $x \in (0, 1)$. Hence, f is strictly decreasing in $(0, 1)$.

It follows that the equation $f(x) = \lambda/a$ has a unique solution in x. Using this fact together with (2), we conclude that $x_1 = \cdots = x_n$. Taking (3) into account, we get $x_1 = \cdots = x_n = 1/n$.

This stationary point is a point of absolute minimum because when $(x_1, \ldots, x_n)$ approaches the boundary of the region determined by

$$\sum_{k=1}^{n} x_k = 1 \quad \text{and} \quad x_k \geq 0 \quad \text{for} \quad k = 1, \ldots, n,$$

then

$$\sum_{k=1}^{n}\left(x_k + \frac{1}{x_k}\right)^a \to +\infty.$$

Therefore,

$$\sum_{k=1}^{n}\left(x_k + \frac{1}{x_k}\right)^a \geq \sum_{k=1}^{n}\left(\frac{1}{n} + n\right)^a = \frac{(n^2+1)^a}{n^{a-1}}.$$

Remark. This result is due to D. S. Mitrinović and D. Ž. Djoković.

If $a = 2$ and $n = 2$, we have the inequality

$$\left(x_1 + \frac{1}{x_1}\right)^2 + \left(x_2 + \frac{1}{x_2}\right)^2 \geq \frac{25}{2} \quad (x_1 > 0, x_2 > 0, x_1 + x_2 = 1)$$

given in: G. H. Hardy: A Course of Pure Mathematics. Cambridge 1948, p. 34.

3.6.42 If $0 < a \leq b$, $a + b = 1$, $0 < y \leq 1$ and $0 < x - y \leq 1$, then

$$\frac{a}{1+x} + \frac{b}{1+y} \leq \frac{1}{1 + x^a y^b}.$$

Reference

Borwein, D.: Solution of Problem 5333. Amer. Math. Monthly **73**, 1023–1024 (1966).

3.6.43 If r is a positive real number and if s is a real number greater than 1, then

$$r\left(\frac{1}{r^s} - \frac{1}{(r+1)^s}\right) < \frac{s}{s-1}\left(\frac{1}{r^{s-1}} - \frac{1}{(r+1)^{s-1}}\right).$$

This inequality is due to D. S. MITRINOVIĆ. For a proof of A. LUPAŞ, see Problem 106. Mat. Vesnik **5 (20)**, 249 (1968).

3.6.44 If $a_1, \ldots, a_n$ are distinct positive numbers, then

$$a_1^{a_1} \cdots a_n^{a_n} > a_1^{a_{i_1}} \cdots a_n^{a_{i_n}}, \tag{1}$$

where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$ and $(i_1, \ldots, i_n) \neq (1, \ldots, n)$.

Proof. Every permutation is a product of disjoint cycles. Therefore, we can assume, without loss of generality, that $i_1, \ldots, i_n$ is a cyclic permutation of $1, \ldots, n$. By renumbering the numbers a_k, we can suppose that $a_1 > a_k$ $(k > 1)$, $i_1 = 2, \ldots, i_{n-1} = n$, $i_n = 1$. Inequality (1) becomes

$$a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n} > a_1^{a_2} a_2^{a_3} \cdots a_n^{a_1}. \tag{2}$$

If $n = 2$, (2) becomes

$$a_1^{a_1} a_2^{a_2} > a_1^{a_2} a_2^{a_1}, \quad \text{i.e.,} \quad a_1^{a_1 - a_2} > a_2^{a_1 - a_2}.$$

This is true since $a_1 > a_2$.

Using induction, we can suppose that

$$a_2^{a_2} \cdots a_n^{a_n} > a_2^{a_3} \cdots a_{n-1}^{a_n} a_n^{a_2}.$$

Multiplying by $a_1^{a_1}$ we get

$$a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n} > a_1^{a_1} a_2^{a_3} \cdots a_{n-1}^{a_n} a_n^{a_2}. \tag{3}$$

Since

$$a_1^{a_1} a_n^{a_2} > a_1^{a_2} a_n^{a_1} \quad (a_1 > a_n \quad \text{and} \quad a_1 > a_2),$$

(3) implies that (2) is true.

3.6.45 If a, b and $x_1, \ldots, x_n$ are positive numbers and if $b^n = x_1 \cdots x_n$, then, for all real t,

$$\sum_{i=1}^{n} \frac{1}{(a + x_i)^t} \lesseqgtr \frac{n}{(a + b)^t}$$

according as $x_i \gtreqless a/t$ $(i = 1, \ldots, n)$.

Reference

KESAVA MENON, P.: Problem 1817. Math. Student **11**, 64 (1943).

3.6.46 Let $a_k > 0$ $(k = 1, \ldots, n)$ and $p_\nu > 0$ $(\nu = 1, \ldots, r)$ with $a_{n+s} = a$ for $s = 1, \ldots, r$. Then

$$\sum_{k=1}^{n} a_k^{p_1 + \cdots + p_r} \geq \sum_{k=1}^{n} a_k^{p_1} \cdots a_{k+r-1}^{p_r}.$$

Proof. This inequality is equivalent to

$$(1) \qquad \sum_{k=1}^{n}\left(p_1 a_k^{\Sigma p} + \cdots + p_r a_{k+r-1}^{\Sigma p} - (\Sigma p)\, a_k^{p_1} \cdots a_{k+r+1}^{p_r}\right) \geq 0,$$

where $\Sigma p = p_1 + \cdots + p_r$, since

$$\sum_{k=1}^{n} a_k^{\Sigma p} = \cdots = \sum_{k=1}^{n} a_{k+r-1}^{\Sigma p}.$$

By virtue of the inequality

$$\frac{p_1 a_1^{\Sigma p} + \cdots + p_r a_r^{\Sigma p}}{\Sigma p} \geq \left(a_1^{p_1 \Sigma p} \cdots a_r^{p_r \Sigma p}\right)^{1/\Sigma p} = a_1^{p_1} \cdots a_r^{p_r},$$

inequality (1) is proved.

The above result is due to D. S. Mitrinović and D. Ž. Djoković.

3.6.47 Let a_{ij} $(i = 1, \ldots, m; j = 1, \ldots, n)$ be nonnegative real numbers not all equal to zero and let $0 < r < s < +\infty$. Then

$$\frac{\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{s}\right)^{r/s}\right)^{1/r}}{\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^{r}\right)^{s/r}\right)^{1/s}} \leq \left(\min(m, n)\right)^{1/r - 1/s}.$$

The constant on the right is the best possible.

Reference

Tôyama, H.: On the inequality of Ingham and Jessen. Proc. Japan Acad. **24**, No. 9, 10–12 (1948).

3.6.48 If $x_1, \ldots, x_n$ are positive numbers and

$$(1) \qquad x_1 + \cdots + x_n = nx,$$

then

$$(2) \qquad \Gamma(x_1) \cdots \Gamma(x_n) \geq \Gamma(x)^n,$$

where Γ denotes the gamma function.

Proof. We shall use the formula (see, for example, [1], p. 250):

$$\frac{d^2}{dz^2} \log \Gamma(z) = \sum_{n=0}^{+\infty} \frac{1}{(z+n)^2}.$$

Hence, $z \mapsto \log \Gamma(z)$ is a convex function in the interval $(0, +\infty)$. We may apply Jensen's inequality

$$\log \Gamma(x_1) + \cdots + \log \Gamma(x_n) \geq n \log \Gamma(x),$$

which is equivalent to (2).

Since

$$\Gamma(x) \leq \sqrt[n]{\Gamma(x_1) \cdots \Gamma(x_n)} \leq \frac{1}{n}\big(\Gamma(x_1) + \cdots + \Gamma(x_n)\big),$$

we have

$$x_1! + \cdots + x_n! \geq nx!, \tag{3}$$

where x and $x_1, \ldots, x_n$ are natural numbers with $x_1 + \cdots + x_n = nx$. Equality holds in (3) if and only if $x_1 = \cdots = x_n = x$.

The above proof is due to D. Ž. Djoković and P. M. Vasić.

Reference

1. Whittaker, E. T., and G. N. Watson: A Course of Modern Analysis. 4th ed., Cambridge 1927.

3.6.49 If m is a positive integer, then

$$\left(m + \frac{1}{4} + \frac{1}{32m + 32}\right)^{1/2} < \frac{\Gamma(m+1)}{\Gamma\left(m + \frac{1}{2}\right)} < \frac{m + \frac{1}{2}}{\left(m + \frac{3}{4} + \frac{1}{32m + 48}\right)^{1/2}}.$$

Reference

Boyd, A. V.: Note on a paper by Uppuluri. Pacific J. Math. **22**, 9–10 (1967).

3.6.50 If $\dfrac{\Gamma\left(x + \frac{1}{2}\right)}{\Gamma(x+1)} = (x + \theta(x))^{-1/2}$, then

$$\frac{1}{4} \leq \theta(x) \leq \frac{1}{2} \quad \text{for} \quad x \geq -\frac{1}{2} \quad \text{and} \quad \frac{1}{4} \leq \theta(x) \leq \frac{1}{\pi} \quad \text{for} \quad x \geq 0.$$

Reference

Watson, G. N.: A note on gamma functions. Proc. Edinburgh Math. Soc. **(2) 11** (1958/59), Edinburgh Math. Notes **42**, 7–9 (1959).

3.6.51 If n is a natural number and $0 \leq s \leq 1$, then

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s}.$$

These inequalities are due to W. Gautschi [1]. A strengthened upper bound for $\Gamma(n+1)/\Gamma(n+s)$ was given by T. Erber [2].

References

1. Gautschi, W.: Some elementary inequalities relating to the gamma and incomplete gamma function. J. Math. and Phys. **38**, 77–81 (1959).
2. Erber, T.: The gamma function inequalities of Gurland and Gautschi. Skand. Aktuarietidskr. **1960**, 27–28 (1961).

3.6.52 If $H_n = \sum_{k=1}^{n} \frac{1}{x+k-1}$ and $G_n = \prod_{k=1}^{n} \left(1 + \frac{x-1}{k}\right)$, then

$$-\frac{\Gamma'(x)}{\Gamma(x)} < H_p + H_q - H_{pq} \leq \frac{1}{x} \quad \text{for} \quad x \geq \frac{1}{2},$$

$$\frac{1}{\Gamma(x)} > \frac{G_p G_q}{G_{pq}} \geq x \text{ for } 0 < x < 1, \text{ and } \frac{1}{\Gamma(x)} < \frac{G_p G_q}{G_{pq}} \leq x \text{ for } x > 1.$$

Remark. For the case $x = 1$ the first inequality above has already been included in 3.1.3.

Reference

NANJUNDIAH, T. S.: Problem 5022. Amer. Math. Monthly **70**, 575–576 (1963).

3.6.53 If b and c are real numbers such that $c > 0$ and $c - 2b > 0$, then

$$\text{(1)} \qquad \frac{\Gamma(c-2b)\,\Gamma(c)}{\Gamma(c-b)^2} \geq \frac{b^2+c}{c},$$

with equality holding if $b = 0$ or $b = -1$.

Remark. Inequality (1) is due to J. GURLAND [1] and it improves and generalizes the WALLIS inequality.

D. GONKALE [2] demonstrated the following inequality

$$\text{(2)} \qquad \frac{\Gamma(c-2b)\,\Gamma(c)}{\Gamma(c-b)^2} > 1 + \frac{b^2(c-2)}{(c-b-1)^2},$$

where b and c are real numbers such that $c > 2$, $c - 2b > 0$, $b \neq 0$ and $b \neq -1$.

In the case $b < 0$, (1) is stronger than (2). If $b > 0$, inequality (2) is stronger than (1).

H. RUBEN [3] gave the following result, connected to inequality (1): If b and c are real numbers such that $c > 0$ and $c - 2b > 0$, then

$$\text{(3)} \qquad \frac{\Gamma(c-2b)\,\Gamma(c)}{\Gamma(c-b)^2} \geq \sum_{k=0}^{n-1} \frac{1}{k!} \frac{((b)_k)^2}{(c)_k},$$

where $(x)_y$ denotes $\frac{\Gamma(x+y)}{\Gamma(x)}$.

Equality holds in (3) if $b = 0, -1, \ldots, -(n-1)$.

Another extension of (1) was given by I. OLKIN [4].

A. V. BOYD [5] has given a simplified proof of (1).

References

1. GURLAND, J.: An inequality satisfied by the gamma function. Skand. Aktuarietidskr. **39**, 171–172 (1956).
2. GONKALE, D.: On an inequality for gamma functions. Skand. Aktuarietidskr. **1962**, 213–215 (1963).
3. RUBEN, H.: Variance bounds and orthogonal expansions in Hilbert space with an application to inequalities for gamma functions and π. J. Reine Angew. Math. **225**, 147–153 (1967).
4. OLKIN, I.: An inequality satisfied by the gamma function. Skand. Aktuarietidskr. **1958**, 37–39 (1959).
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3.6.54 Let n be a positive integer and let $c \geq -1$ be independent of n. Then

$$\frac{(2n-1)!!}{(2n)!!}\sqrt{n+c} < \frac{1}{\sqrt{\pi}} \quad \text{for} \quad c \leq \frac{1}{4},$$

$$> \frac{1}{\sqrt{\pi}} \quad \text{for} \quad c \geq \frac{n+1}{4n+3}.$$

For all integers $n \geq 2$,

$$\text{(1)} \qquad \sqrt{\frac{2n-3}{2n-2}} < \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} < \sqrt{\frac{2n-2}{2n-1}}.$$

Furthermore, for all odd integers $n \geq 3$, the first member in (1) can be replaced by

$$\frac{n}{2}\sqrt{(2n-2)(2n+1)}.$$

There exist also the following estimates

$$\sqrt{\frac{4n-3}{4n-2}} < 2^{2n-1}\sqrt{\frac{2n-1}{2\pi}}\, B(n, n) < \sqrt{\frac{4n-2}{4n-1}},$$

where $n = 1, 2, \ldots$ and B is the beta function.

Reference

CHU, J. T.: A modified Wallis product and some applications. Amer. Math. Monthly **69**, 402—404 (1962).

3.6.55 If $x > 1$, then

$$0 < \log \Gamma(x) - \left[\left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi)\right] < \frac{1}{x},$$

$$\frac{1}{2x} < \log x - \frac{\Gamma'(x)}{\Gamma(x)} < \frac{1}{x}$$

and

$$\frac{1}{x} < \frac{d^2}{dx^2}\log \Gamma(x) < \frac{1}{x-1},$$

where Γ denotes the gamma function.

Remark. The function g, defined by

$$g(x) = x\,\frac{\Gamma(x+2)^{\frac{1}{x+1}}}{\Gamma(x+1)^{\frac{1}{x}}},$$

is strictly concave for $x \geq 6$.

References

MINC, H., and L. SATHRE: Some inequalities involving $(r!)^{1/r}$. Proc. Edinburgh Math. Soc. (2) 14, 41–46 (1964/65).

BASS, J.: Cours de mathématiques, vol. 1. Paris 1961, p. 548.

3.6.56 If $n = 1, 2, \ldots$ and $0 < r < 1$, then

$$n^r - (n-1)^r \geq \frac{r}{n!} \Gamma(n+r).$$

Reference

ZIMERING, SH.: On a Mercerian theorem and its applications to the equiconvergence. Publ. Inst. Math. Beograd 1 (15), 88–89 (1961).

3.6.57 Let functions G and H be defined by

$$G(u) = e^u u^{-u} \Gamma(1+u), \quad G(0) = 1,$$

and

$$H(u, v) = \frac{G(u+v)}{G(u)\,G(v)} = \frac{u^u v^v}{(u+v)^{u+v}} \cdot \frac{\Gamma(1+u+v)}{\Gamma(1+u)\,\Gamma(1+v)}.$$

Then, for $a > 0$ and $b \geq 1$,

$$\frac{1}{e} < \frac{1}{G\left(\frac{b-1}{b}\right)} \leq H\left(\frac{1}{a}, \frac{b-1}{b}\right) \leq 1,$$

$$\frac{1}{e} < \frac{1}{(1+a)^{1/a}} < H\left(\frac{1}{a}, \frac{b-1}{b}\right) \leq 1.$$

Reference

SCHMIDT, E.: Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet. Math. Ann. 117, 301–326 (1940).

3.6.58 If $p > 0$, $q > 0$, $p > r$ and $q > s$, then for $r > 0$ and $s > 0$, we have

$$B(p, q) \leq \left(\frac{r}{r+s}\right)^r \left(\frac{s}{r+s}\right)^s B(p-r, q-s),$$

where B is the beta function.

For $r < 0$ and $s < 0$, the reverse inequality holds.

Reference

KESAVA MENON, P.: Some inequalities involving the Γ- and ζ-functions. Math. Student 11, 10–12 (1943).

3.7 Integral Inequalities

In many Sections inequalities involving integrals were treated. In this Section we also give some unconnected integral inequalities which are more or less general. However, our aim was mainly to collect those which appear to be simple, but interesting, and which at the same time seem to be useful in various research.

3.7.1 Let x, y, z be positive numbers not all equal, and define

$$R = \frac{1}{2}\int_0^{+\infty} ((t + x^2)(t + y^2)(t + z^2))^{-1/2}\,dt;$$

$$\alpha = \frac{3}{\sum \frac{yz}{x}}, \quad \beta = \left(\frac{3}{\sum x^2}\right)^{1/2}, \quad \gamma = \left(\frac{3}{\sum xy}\right)^{1/2}, \quad \delta = \frac{3}{\sum (xy)^{1/2}},$$

$$\varepsilon = (xyz)^{-1/3}, \quad \xi = \frac{1}{3}\sum \frac{1}{x}, \quad \eta = \left(\frac{1}{3}\sum \frac{1}{x^2}\right)^{1/2}, \quad \theta = \frac{1}{3}\sum \frac{x}{yz},$$

where Σ denotes a summation over the three cyclic permutations of x, y, z.

Let $\alpha_n, \ldots, \theta_n$ denote the result of replacing x, y, z in the expressions for $\alpha, \ldots, \theta$ by x_n, y_n, z_n, where

$$x_{n+1} = \frac{1}{2}(x_n + y_n)^{1/2}(x_n + z_n)^{1/2}, \quad y_{n+1} = \frac{1}{2}(y_n + z_n)^{1/2}(y_n + x_n)^{1/2},$$

$$z_{n+1} = \frac{1}{2}(z_n + x_n)^{1/2}(z_n + y_n)^{1/2} \quad (n = 0, 1, \ldots)$$

and $x_0 = x$, $y_0 = y$, $z_0 = z$. Then, for $n > 2$,

$$\alpha < \beta < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < R,$$

$$R < \xi_n < \eta_n < \theta_n < \cdots < \xi_1 < \eta_1 < \theta_1 < \xi < \eta < \theta,$$

$$R < \gamma_n < \delta_n < \varepsilon_n < \xi_n < \eta_n < \cdots < \gamma_1 < \delta_1 < \varepsilon_1 < \xi_1 < \eta_1 < \gamma < \delta < \varepsilon < \xi < \eta.$$

This result was privately communicated to us by B. C. Carlson [1], and it contains the results obtained earlier by G. Pólya and G. Szegö [2], G. N. Watson [3], G. Szegö [4], G. Pólya and G. Szegö [5].

References

1. Carlson, B. C.: Inequality for a symmetric elliptic integral. Proc. Amer. Math. Soc. (to appear).
2. Pólya, G., and G. Szegö: Problem 542. Arch. Math. Phys. **(3) 28**, 81–82 (1919).
3. Watson, G. N.: An inequality. Quart. J. Math. Oxford Ser. **5**, 221–223 (1934).
4. Szegö, G.: An inequality. Quart. J. Math. Oxford Ser. **6**, 78–79 (1935).
5. Pólya, G., and G. Szegö: Inequalities for the capacity of a condenser. Amer. J. Math. **67**, 1–32 (1945).

3.7.2 Real polynomials $P_1, P_2, \ldots$, defined recursively by

$$P_n(x) = 1 + \int_0^x P_{n-1}(t - t^2)\,dt \quad (n = 1, 2, \ldots), \text{ with } P_0(x) = 1,$$

satisfy

$$0 \le P_n(x) - P_{n-1}(x) \le \frac{x^n}{n!} \text{ for } 0 \le x \le 1.$$

Reference

Problem 16346. Revue Math. Spéc. **76**, 221 (1965/66).

3.7.3 If $a > 0$, then

$$\int_a^{+\infty} e^{-x^2}\, dx < \min\left(\frac{\sqrt{\pi}}{2}\, e^{-a^2}, \frac{1}{2a}\, e^{-a^2}\right).$$

Reference

Ostrowski 3, p. 303.

3.7.4 Let f be defined by $f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Then for $x \geq 0$ and $y \geq 0$ we have

$$(1) \qquad f(x)\, f(y) \geq f(x) + f(y) - f(x+y),$$

with equality if and only if x or y is an end point of the closed interval $[0, +\infty]$.

Proof. First note that $f(0) = 0$ and $f(+\infty) = 1$. Then for any fixed $y > 0$, the function F defined for $x \geq 0$ by

$$F(x) = f(x)\, f(y) - f(x) - f(y) + f(x+y)$$

satisfies $F(0) = F(+\infty) = 0$. Rolle's theorem applied to the interval $[0, +\infty]$ insures the existence of a t in $(0, +\infty)$ such that $F'(t) = 0$. The derivative F' given by

$$F'(x) = \frac{2}{\sqrt{\pi}}\, e^{-x^2}\left(f(y) - 1 + e^{-2xy-y^2}\right)$$

is a decreasing function for $x > 0$. It follows that such a t is unique and F' has the same sign as $t - x$. Therefore 0 and $+\infty$ are the only zeros of F and F is positive elsewhere.

The above proof is due to R. J. Weinacht.

Reference

Mitrinović, D. S.: Problem 5555. Amer. Math. Monthly **75**, 84 and 1129–1130 (1968).

Remark. In Mitrinović 1, pp. 132–133, a proof of the inequality

$$(2) \qquad h(-x-y) \leq 2h(-x)\, h(-y)$$

for $x \geq 0$ and $y \geq 0$ is given, where

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

which is similar to (1).

Inequality (2) is due to C. G. Esseen [see Nord. Mat. Tidskr. **9**, 137 (1961)].

3.7.5 For all real x,

$$\int_0^x e^{-x^2/2}\,dx > \sqrt{\frac{\pi}{2}}\left(1 - e^{-x^2} - \frac{1}{2}e^{-x^2/2}\right).$$

Reference

KRISHNASWAMI AYYANGAR, A. A.: Problem 1800. Math. Student 11, 63 (1943).

3.7.6 If $x > 0$, then

$$\int_0^{\pi/2} e^{-x^2\sin^2 t}\sin t\,dt \leq \frac{\pi^2}{8x^2}\left(1 - e^{-x^2}\right).$$

3.7.7 Let $f(n) = \int_0^{\pi/4} \tan^n x\,dx$ (n is a natural number). Then

$$f(n+1) < f(n) \quad \text{for } n > 1,$$

$$\frac{1}{n+1} < 2f(n) < \frac{1}{n-1} \quad \text{for } n > 2.$$

3.7.8 If n is a natural number, then

$$\frac{\pi}{2(n+1)} < \left(\int_0^{\pi/2} \sin^n\theta\,d\theta\right)^2 < \frac{\pi}{2n}.$$

3.7.9 If $x \geq 0$ and $p > 1$, then

$$\frac{2}{\pi}\int_x^{xp}\left(\frac{\sin t}{t}\right)^2 dt \leq 1 - \frac{1}{p}.$$

Reference

MASANI, P. R.: Problem 134. Nieuw Arch. Wisk. (3) 15, 172–173 (1967).

3.7.10 For $x > 0$,

$$\left|\int_x^{x+1} \sin(t^2)\,dt\right| < \frac{2}{x}.$$

3.7.11 If n denotes a positive integer and λ a positive number, then

$$\frac{\pi}{(1 + (n+1)^\lambda \pi^\lambda)^{1/2}} < \int_{n\pi}^{(n+1)\pi} \frac{1}{1 + x^\lambda \sin^2 x}\,dx < \frac{\pi}{(1 + n^\lambda \pi^\lambda)^{1/2}}.$$

From University examination papers, The University of Glasgow 1958.

Comment. A proof of these inequalities as well as some stronger ones can be found in: D. S. MITRINOVIĆ: Zbornik matematičkih problema, vol. 3. Beograd 1960, p. 267. Those stronger inequalities are:

$$\frac{\pi}{2}\left\{\frac{1}{\left[1 + \left(n + \frac{1}{2}\right)^\lambda \pi^\lambda\right]^{1/2}} + \frac{1}{[1 + (n+1)^\lambda \pi^\lambda]^{1/2}}\right\} < \int_{n\pi}^{(n+1)\pi} \frac{1}{1 + x^\lambda \sin^2} dx$$

$$< \frac{\pi}{2}\left\{\frac{1}{(1 + n^\lambda \pi^\lambda)^{1/2}} + \frac{1}{\left[1 + \left(n + \frac{1}{2}\right)^\lambda \pi^\lambda\right]^{1/2}}\right\}.$$

3.7.12 If $0 < t < \pi$ and

$$I(t) = \int_0^1 \frac{1}{(1 - 2x\cos t + x^2)^{3/2}}\, dx,$$

then

(1) $$I(t) < \frac{\pi^2(\pi - t)}{8t^2} \qquad \left(0 < t \leq \frac{\pi}{2}\right),$$

(2) $$I(t) < \frac{1}{\sin^3 t} \qquad (0 < t < \pi).$$

Proof. Substituting $x = \cos t + y \sin t$, we get

(3) $$I(t) = \frac{1}{\sin^2 t} \int_a^b \frac{1}{(1 + y^2)^{3/2}} dy \left(\text{where } a = \tan\left(t - \frac{\pi}{2}\right) \text{ and } b = \tan\frac{t}{2}\right)$$
$$= \frac{1}{\sin^2 t} \sin(\arctan y)\Big|_a^b$$
$$= \frac{1}{\sin^2 t}\left[\sin\frac{t}{2} - \sin\left(t - \frac{\pi}{2}\right)\right].$$

(3) together with

$$\sin x - \sin y < |x - y| \qquad (x \neq y),$$

and

$$\sin t > \frac{2}{\pi} t \qquad \left(0 < t < \frac{\pi}{2}\right)$$

implies (1).

Inequality (2) can be proved as follows:

$$(x - \cos t)^2 + \sin^2 t \geq \sin^2 t,$$
$$1 - 2x\cos t + x^2 \geq \sin^2 t,$$
$$\frac{1}{(1 - 2x\cos t + x^2)^{3/2}} \leq \frac{1}{\sin^3 t}.$$

By integration over $(0, 1)$ we obtain (2).

The proof of inequality (2) is due to B. Mesihović.

Remark. Substituting (3) into (2), we get an interesting inequality

$$\sin\frac{t}{2} + \cos t < \frac{1}{\sin t} \qquad (0 < t < \pi).$$

It is easy to prove this inequality by dividing the interval $(0, \pi)$ into two subintervals $(0, \pi/3)$, $(\pi/3, \pi)$.

See also **9.14 in Mitrinović 1**, where is proved that

$$I(t) < \frac{\pi^2}{2t^2} \quad \text{for} \quad 0 < t \leq \frac{\pi}{2},$$

and this result is weaker than (1).

3.7.13 The following inequalities hold:

$$\int_0^1 \tanh \frac{1}{ax}\, dx < 1 \quad \text{for} \quad 0 < a \leq 1 \tag{1}$$

and

$$\int_0^1 \tanh \frac{1}{ax}\, dx < \frac{1 + \log a}{a} \quad \text{for} \quad a > 1. \tag{2}$$

Proof. Since $\tanh x < 1$ and $\tanh x < x$ for $x > 0$, we have

$$\tanh \frac{1}{ax} < \min\left(1, \frac{1}{ax}\right) \quad \text{for} \quad a > 0 \quad \text{and} \quad x > 0,$$

and therefore

$$\int_0^1 \tanh \frac{1}{ax}\, dx < \int_0^1 \min\left(1, \frac{1}{ax}\right) dx \quad (a > 0),$$

from where we get (1) and (2).

Revised from a proof by D. V. SLAVIĆ.

3.7.14 Let n be a natural number, t a real number and $f(t) = \prod_{k=1}^{n-1} (t - k)$. Then

$$\frac{1}{(n-1)!} \int_n^{+\infty} e^{-t} f(t)\, dt < \left(\frac{2}{e}\right)^n.$$

Reference

OLDS, C. D.: Problem 4198. Amer. Math. Monthly **53**, 225 (1946).

3.7.15 If $a_1, \ldots, a_n$ are real numbers, then

$$\int_0^{+\infty} e^{-x} (1 + a_1 x + \cdots + a_n x^n)^2\, dx \geq \frac{1}{n+1}.$$

Remark. This inequality was posed as a problem in book [1] by F. BOWMAN and F. A. GERARD. A simple proof was given by L. J. MORDELL in [2]. This inequality was later generalized by L. J. MORDELL [3] who determined the minimum value of some integrals of the form

$$\int_p^q f(x)\, (a_0 + a_1 x + \cdots + a_n x^n)^2\, dx,$$

where f is a real function such that for $r \geq 0$ the integral $\int_p^q f(x)\, x^r\, dx$ exists.

References

1. BOWMAN, F., and F. A. GERARD: Higher Calculus. Cambridge 1967, p. 327.
2. MORDELL, L. J.: The minimum value of a definite integral. Math. Gaz. **52**, 135–136 (1968).
3. MORDELL, L. J.: The minimum value of a definite integral II. Aequationes Math. **2**, 327–331 (1969).

3.7.16 Let P_n be a real polynomial of degree n with the property $P_n(x) \geq 0$ for $x \geq 0$. Then

$$-\left[\frac{n}{2}\right] \int_0^{+\infty} P_n(x)\, e^{-x}\, dx \leq \int_0^{+\infty} P'_n(x)\, e^{-x}\, dx \leq \int_0^{+\infty} P_n(x)\, e^{-x}\, dx.$$

Equality holds in the first inequality if and only if $P_n(x)$ is a non-negative multiple of $\left(\sum_{j=0}^{[n/2]} L_j(x)\right)^2$, where $L_j(x)$ is the LAGUERRE polynomial of degree j. Equality in the second inequality occurs if and only if $P_n(0) = 0$.

Reference

PHILLIPS, D. L.: Remark on an inequality of Shampine. Amer. Math. Monthly **75**, 511–512 (1968).

3.7.17 If f and g are real integrable functions on $[a, b]$, then

$$\begin{aligned}\int_a^b \min(f(x), g(x))\, dx &\leq \min\left(\int_a^b f(x)\, dx, \int_a^b g(x)\, dx\right)\\ &\leq \max\left(\int_a^b f(x)\, dx, \int_a^b g(x)\, dx\right)\\ &\leq \int_a^b \max(f(x), g(x))\, dx.\end{aligned}$$

Communicated by P. R. BEESACK.

3.7.18 Let $f_0(x) > 0$ for $x \geq 0$ and let $f_n(x) = \int_0^x f_{n-1}(t)\, dt$ for $n = 1, 2, \ldots$ Then, for $x > 0$ and $n = 0, 1, 2, \ldots,$

$$(1) \qquad n f_{n+1}(x) < x f_n(x).$$

Proof. The result is obvious for $n = 0$. If we assume that (1) holds for some n, then for $x > 0$,

$$n f_{n+2}(x) = n \int_0^x f_{n+1}(t)\, dt < \int_0^x t f_n(t)\, dt = x f_{n+1}(x) - f_{n+2}(x),$$

which implies

$$(n + 1)\, f_{n+2}(x) < x f_{n+1}(x) \quad \text{for} \quad x > 0.$$

This completes the induction proof.

The above result is due to ROY O. DAVIES.

3.7.19 Let f be a continuous nonnegative function for $x > 0$ and let

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$

If g is defined by

$$g(t) = \int_{-\infty}^{+\infty} f(x) \cos tx\,dx \quad (t \text{ real}),$$

then

$$g(2t) > 2g(t)^2 - 1 \quad \text{for } t \neq 0.$$

Remark. A proof of this inequality can be found in: MITRINOVIĆ 1, pp. 133–134.

3.7.20 Let f be a positive continuous function on $[0, 1]$ and

$$I_n = \int_0^1 f(x)^n dx;$$

then, for $n > 1$,

$$I_{n-1}^2 \leq I_n I_{n-2}.$$

Proof. Applying the BUNIAKOWSKI-SCHWARZ inequality, we obtain

$$I_{n-1}^2 = \left(\int_0^1 f(x)^{n-1}\,dx\right)^2 = \left(\int_0^1 f(x)^{\frac{n}{2}} f(x)^{\frac{n-2}{2}}\,dx\right)^2$$
$$\leq \int_0^1 f(x)^n\,dx \int_0^1 f(x)^{n-2}\,dx = I_n I_{n-2}.$$

Reference

Problem 19241. Revue Math. Spéc. **75**, 473 (1964/65).

3.7.21 If the integrals

$$\int_a^b F(x)^2\,dx \quad \text{and} \quad \int_a^b (F(x) - x^r)^2\,dx \ (\equiv \lambda^2)$$

exist, then

(1) $$\left[|\lambda| - \left(\frac{b^{2r+1} - a^{2r+1}}{2r+1}\right)^{1/2}\right]^2 \leq \int_a^b F(x)^2\,dx$$
$$\leq \left[|\lambda| + \left(\frac{b^{2r+1} - a^{2r+1}}{2r+1}\right)^{1/2}\right]^2.$$

Proof. Let $F(x) = G(x) + H(x)$. If the integrals involved exist, we have

(2) $$\left(\int_a^b F(x)^2\,dx\right)^{1/2} \leq \left(\int_a^b G(x)^2\,dx\right)^{1/2} + \left(\int_a^b H(x)^2\,dx\right)^{1/2}$$

and

$$\left(\int_a^b (F(x) - G(x))^2\,dx\right)^{1/2} \leq \left(\int_a^b F(x)^2 dx\right)^{1/2} + \left(\int_a^b G(x)^2\,dx\right)^{1/2},$$

i.e.,

$$\left(\int_a^b F(x)^2\,dx\right)^{1/2} \geq \left(\int_a^b \left(F(x) - G(x)\right)^2 dx\right)^{1/2} - \left(\int_a^b G(x)^2\,dx\right)^{1/2}. \tag{3}$$

Putting $G(x) = x^r$ in (2) and (3), we get (1).

Reference

Klamkin, M. S., and M. J. Cohen: Problem **543**. Math. Mag. **37**, 281–282 (1964).

3.7.22 Let f be a differentiable real-valued function defined on $[0, 1]$ and such that

$$|f'(x)| \leq M \text{ for } 0 < x < 1.$$

Then

$$\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right| \leq \frac{M}{n}.$$

Reference

Mackey, G. W.: The William Lowell Putnam Mathematical Competition. Amer. Math. Monthly **54**, 403 (1947).

3.7.23 Let f be a differentiable function on (a, b) and let, on (a, b), $|f'(x)| \leq M$. Then, for every $x \in (a, b)$,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\,M.$$

Reference

Ostrowski, A.: Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. Comment. Math. Helv. **10**, 226–227 (1938).

3.7.24 Let f be a differentiable function on $[a, b]$ and let $|f'(x)| \leq M$ $(M > 0)$. Then

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)\left(f(a) + f(b)\right)\right| < \frac{M(b-a)^2}{4} - \frac{1}{4M}\left(f(b) - f(a)\right)^2. \tag{1}$$

Inequality (1) has been proved by K. S. K. Iyengar [1], while the same inequality has been proved geometrically by G. S. Mahajani [2].

Using similar geometrical arguments, G. S. Mahajani [2] has also proved the following results:

1° If f has a bounded derivative on $[a, b]$, i.e., if $|f'(x)| \leq M$ $(M > 0)$ and if $\int_a^b f(x)\,dx = 0$, then for $x \in [a, b]$,

$$\left|\int_a^x f(x)\,dx\right| \leq \frac{M(b-a)^2}{8}; \tag{2}$$

2° If, besides the conditions given in 1°, $f(a) = f(b) = 0$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{M(b-a)^2}{16}. \tag{3}$$

Remark 1. Putting $f(a) = f(b) = 0$ in (1), we obtain an inequality which is equivalent to the one indicated in Problem 121 of [3].

Remark 2. Putting $g(x) = \int_a^x f(t)\,dt$, from 1° and 2°, we obtain the following results:

If $|g''(x)| \leq M$ on $[a, b]$ and if $g(b) = 0$, then

$$|g(x)| \leq \frac{M(b-a)^2}{8}.$$

If $|g''(x)| \leq M$ on $[a, b]$ and if $g(b) = g'(a) = g'(b) = 0$, then

$$|g(x)| \leq \frac{M(b-a)^2}{16}.$$

Comment by P. R. Beesack. Inequality (1) of K. S. K. Iyengar appears to be a very good one. A related inequality which is very easy to prove is

$$m\,\frac{(b-a)^2}{2} \leq \int_a^b f(x)\,dx - (b-a)\,f(a) \leq M\,\frac{(b-a)^2}{2},$$

if f' is continuous on $[a, b]$, and $m \leq f'(x) \leq M$ on $[a, b]$.

To prove this, just integrate $\int_a^b (f(x) - f(a))\,dx$ by parts.

Remark 3. Let f have a continuous $2n$-th derivative on $[a, b]$ and let $|f^{(2n)}(x)| \leq M$ and $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r = 0, 1, \ldots, n-1$. Then

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(n!)^2\,M}{(2n)!\,(2n+1)!}\,(b-a)^{2n+1}.$$

This inequality (see [4]) is in some connection with the above results.

References

1. Iyengar, K. S. K.: Note on an inequality. Math. Student **6**, 75–76 (1938).
2. Mahajani, G. S.: A Note on an inequality. Math. Student **6**, 125–128 (1938).
3. Pólya, G., and G. Szegö: Aufgaben und Lehrsätze aus der Analysis, vol. 1. Berlin 1925, p. 62, Problem 121.
4. Problem E 2155. Amer. Math. Monthly **76**, 188 and 1142–1143 (1969).

3.7.25 Let $n > m > 1$, and let f be a real nonnegative function on $[0, +\infty)$ such that the integral $\int_0^{+\infty} f(x)^m\,dx$ exists in the Lebesgue sense.

Then the integral $y = \int_0^x f(x)\,dx$ is finite for every x and

$$\int_0^{+\infty} \frac{y^n}{x^{n-r}}\,dx \leq \frac{1}{n-r-1}\left(\frac{r\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{n-1}{r}\right)}\right)^r \left(\int_0^{+\infty} f(x)^m\,dx\right)^{n/m},$$

where $r = \frac{n}{m} - 1$.

Reference

Bliss, G. A.: An integral inequality. J. London Math. Soc. **5**, 40–46 (1930).

3.7.26 Let $p > 1$, $f(x) \geq 0$, $r(x) > 0$ for $x > 0$, and let r be an absolutely continuous function. If for some $\lambda > 0$ and for almost all x

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda},$$

and if

$$H(x) = \frac{1}{xr(x)} \int_0^x r(t)\,f(t)\,dt,$$

then

$$\int_0^{+\infty} H(x)^p\,dx \leq \lambda^p \int_0^{+\infty} f(x)^p\,dx.$$

Remark. The above result together with some similar ones can be found in article [1] by N. Levinson. For $r(x) \equiv 1$ and $\lambda = \frac{p}{p-1}$ the inequality quoted in [2], p. 240, is obtained.

References

1. Levinson, N.: Generalizations of an inequality of Hardy. Duke Math. J. **31**, 389–394 (1964).
2. Hardy, Littlewood, Pólya: Inequalities.

3.7.27 Let f be a function which has a continuous first derivative on $[a, b]$. Then, for $a \leq x \leq b$,

$$\left|\left(f(x) - \frac{f(a)+f(b)}{2}\right)^2 - \left(\frac{f(b)-f(a)}{2}\right)^2\right|$$
$$\leq \left[\int_a^b \left(f(x) - \frac{f(a)+f(b)}{2}\right)^2 dx \int_a^b f'(x)^2\,dx\right]^{1/2}.$$

Reference

Ostrowski 3, p. 360.

3.7.28 If f is a periodic function with the primitive period 2π, and if f' is continuous, then, for every x and y,

$$|f(x)^2 - f(y)^2| \leq \int_0^{2\pi} f(x)^2\,dx \int_0^{2\pi} f'(x)^2\,dx.$$

This result is due to S. Warschawski (see: Ostrowski 3, p. 360).

3.7.29 If the second derivative f'' of a real function f is a continuous function on $[a, b]$, then

$$\int_a^b (f''(x))^2\,dx \geq \frac{12}{(b-a)^3}\left[f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right]^2.$$

Reference

ZMOROVIČ, V. A.: On some inequalities (Russian). Izv. Polytehn. Inst. Kiev **19**, 92–107 (1956).

3.7.30 If f is never increasing for $x > 0$, then, for any $\lambda > 0$,

$$\lambda^2 \int_\lambda^{+\infty} f(x)\,dx \leq \frac{4}{9}\int_0^{+\infty} x^2 f(x)\,dx.$$

This inequality is due to GAUSS. See [1], and for a generalization, [2].

References

1. CRAMÉR, H.: Methods of Mathematical Statistics. Princeton **1946**, p. 256, Ex. 4.

2. VOLKOV, V. N.: Inequalities connected with an inequality of Gauss (Russian). Volž. Mat. Sb. 1969, No. 7, 11–13.

3.7.31 Let f be a positive and nonincreasing function over the interval $[1, +\infty)$. If

$$g_n(t) = t^n f(t^n) \quad \text{for} \quad t > 1 \quad \text{and} \quad n = 0, 1, \ldots,$$

then

$$\frac{t-1}{t}\, g_{n+1}(t) \leq \int_{t^n}^{t^{n+1}} f(x)\,dx \leq (t-1)\, g_n(t).$$

3.7.32 If $0 < a < b$ and $f(x) \geq 0$, $(xf(x))' \geq 0$, then

$$\left|\int_a^b f(x)\cos(\log x)\,dx\right| \leq 2bf(b).$$

Proof. From $(xf(x))' \geq 0$ it follows that $x \mapsto xf(x)$ is an increasing function. Hence,

$$\begin{aligned}\left|\int_a^b f(x)\cos(\log x)\,dx\right| &= \left|\int_a^b xf(x)\,\frac{\cos(\log x)}{x}\,dx\right| \\ &\leq bf(b)\left|\int_a^b \cos(\log x)\,\frac{dx}{x}\right| \\ &= bf(b)\,|\sin(\log b) - \sin(\log a)| \\ &\leq 2bf(b).\end{aligned}$$

3.7.33 Let g be a monotone and integrable function on $[a, b]$. Then

$$\left|\int_a^b g(x) \cos x \, dx\right| \leq 2 \left(|g(a) - g(b)| + |g(b)|\right).$$

Reference

Ostrowski 3, p. 137.

3.7.34 Let f be a real-valued, continuous function of x on $[a, b]$ which is not identically zero, and which satisfies the condition $0 \leq f(x) \leq M$. Then

$$(1) \qquad 0 < \left(\int_a^b f(x) \, dx\right)^2 - \left(\int_a^b f(x) \cos x \, dx\right)^2 - \left(\int_a^b f(x) \sin x \, dx\right)^2 \leq \frac{1}{12} M^2 (b-a)^4.$$

This result is due to O. Dunkel.

Remark 1. A. A. Bennett has proved a rather extensive generalization of the above inequalities. However, since the formulation of his hypotheses is rather complicated, we do not quote Bennett's generalization.

Remark 2. D. Ž. Djoković gave the following comment on Dunkel's inequality: Let

$$J = \left(\int_a^b f(x) \, dx\right)^2 - \left(\int_a^b f(x) \cos x \, dx\right)^2 - \left(\int_a^b f(x) \sin x \, dx\right)^2.$$

Then

$$J = \iint_D f(x) f(y) (1 - \cos x \cos y - \sin x \sin y) \, dx \, dy$$

$$= \iint_D f(x) f(y) (1 - \cos(x - y)) \, dx \, dy,$$

where D is the square $[a, b] \times [a, b]$.

Therefore,

$$0 < J \leq M^2 \iint_D (1 - \cos(x - y)) \, dx \, dy = M^2 (b-a)^2 \left[1 - \left(\frac{\sin \frac{b-a}{2}}{\frac{b-a}{2}}\right)^2\right].$$

This is the best possible bound for J, and is, therefore, better than (1).

Reference

Dunkel, O., and A. A. Bennett: Problem 3104. Amer. Math. Monthly **32**, 319–321 (1925).

3.7.35 Let g be a monotone and integrable function on $[a, b]$ and let $g(a) g(b) \geq 0$, $|g(a)| \geq |g(b)|$. Let, further, f be a real or a complex function, integrable on $[a, b]$. Then

$$\left|\int_a^b f(x) g(x) \, dx\right| \leq |g(a)| \max_{a \leq u \leq b} \left|\int_a^u f(x) \, dx\right|.$$

Reference

Ostrowski 3, p. 141.

3.7.36 Let g be a monotone and integrable function on $[a, b]$ and let f be a real or a complex function integrable on $[a, b]$. Then

$$\left|\int_a^b f(x)\, g(x)\, dx\right| \leq |g(a)| \max_{a \leq u \leq b} \left|\int_a^u f(x)\, dx\right| + |g(b)| \max_{a \leq u \leq b} \left|\int_u^b f(x)\, dx\right|.$$

Reference

Ostrowski 3, p. 141.

3.7.37 Let $x \mapsto x + f(x)$ be a nondecreasing function on $[-1, +1]$ with $f(1) \leq f(-1)$, and let $\int_{-1}^{+1} f(x)\, dx = 0$. Then

$$\int_{-1}^{+1} f(x)^2\, dx \leq \frac{2}{3}.$$

For $f(x) = -x$, equality holds.

Reference

van Dantzig, D.: Problem 92. Wisk. Opgaven **19**, 196–199 (1950/54).

3.7.38 Let f be a nondecreasing continuous function on $[0, \pi]$, such that $f(0) = 0$ and $f(\pi) = \pi$. Then

$$\left|\int_0^\pi \exp(if(t) - it)\, dt\right| > 2$$

and the constant 2 cannot be replaced by any greater number.

Reference

Ullman, J. L., and C. J. Titus: An integral inequality with application to harmonic mappings. Michigan Math. J. **10**, 181–192 (1963).

3.7.39 If f is a nonnegative and nonincreasing function over $0 \leq x \leq 1$, then

$$\frac{\int_0^1 x f(x)^2\, dx}{\int_0^1 x f(x)\, dx} \leq \frac{\int_0^1 f(x)^2\, dx}{\int_0^1 f(x)\, dx}. \tag{1}$$

Proof. By the above hypotheses we have

$$\int_0^1 \int_0^1 f(x)\, f(y)\, (x - y)\, [f(x) - f(y)]\, dx\, dy \leq 0,$$

i.e.,

$$\left(\int_0^1 xf(x)^2\,dx\right)\left(\int_0^1 f(y)\,dy\right) - \left(\int_0^1 xf(x)\,dx\right)\left(\int_0^1 f(y)^2\,dy\right)$$
$$-\left(\int_0^1 f(x)^2\,dx\right)\left(\int_0^1 yf(y)\,dy\right) + \left(\int_0^1 yf(y)^2\,dy\right)\left(\int_0^1 f(x)\,dx\right)$$
$$= 2\left(\int_0^1 xf(x)^2\,dx\right)\left(\int_0^1 f(x)\,dx\right) - 2\left(\int_0^1 xf(x)\,dx\right)\left(\int_0^1 f(x)^2\,dx\right) \le 0,$$

whence (1) follows.

References

SCHWERDTFEGER, H.: An inequality for monotonic functions. Newsletter of the Canadian Mathematical Congress, 4th Issue, Dec. 1955, pp. 11—13.

BUSH, L. E.: The William Lowell Putnam Mathematical Competition. Amer. Math. Monthly **64**, 650 (1957).

Remark. For $p(x) = F(x)^2$, $f(x) = x$, $g(x) = 1/F(x)$, Theorem 10 in 2.5 yields inequality (1).

3.7.40 Let f be an integrable function for $x \in [a, b]$ and let $F(x) = \int_a^x f(t)\,dt$, $|F(x)| \le M(x - a)$ for $a < x \le b$ (M a positive constant); furthermore, let g be a nonnegative, nonincreasing and integrable function. Then

$$\left|\int_{a+}^b f(x)\,g(x)\,dx\right| \le M\int_a^b g(x)\,dx. \tag{1}$$

Proof. Let $a < \alpha < \beta \le b$; we have

$$\int_a^\alpha g(x)\,dx \ge (\alpha - a)\,g(\alpha).$$

Then

$$\left|\int_\alpha^\beta f(x)\,g(x)\,dx - g(\beta)\int_\alpha^\beta f(x)\,dx\right| = \left|\int_\alpha^\beta f(x)\,[g(x) - g(\beta)]\,dx\right|$$
$$\le |F(\alpha)\,[g(\alpha) - g(\beta)]| + \left|\int_\alpha^\beta F(x)\,d[-g(x) + g(\beta)]\right|$$
$$\le M(\alpha - a)\,g(\alpha) + M\int_\alpha^\beta (x - a)\,d[-g(x) + g(\beta)]$$
$$\le 2M\int_a^\alpha g(x)\,dx + M\int_\alpha^\beta g(x)\,dx - M(\beta - \alpha)\,g(\beta).$$

Letting α and β tend to a, we see that $\int_{a+}^{b} f(x)\, g(x)\, dx$ exists. Taking $\beta = b$ and letting $\alpha \to a+$, we obtain

$$\left| \int_{a+}^{b} f(x)\, g(x)\, dx \right| \leq M \int_{a}^{b} g(x)\, dx + g(b) \left[F(b) - M(b-a)\right],$$

which is somewhat stronger than (1), since $g(b)\,[F(b) - M(b-a)]$ is negative.

Remark. The above is, in fact, a review of R. P. Boas, published in Math. Reviews **9**, 136–137 (1948).

Reference

Natanson, I. P.: On an inequality (Russian). Doklady Akad. Nauk SSSR (N.S.) **56**, 911–913 (1947).

3.7.41 Let a and b be given positive numbers and let f be a real function such that $f(0) = 0$, $f(a) = b$, $f(x) \geq 0$ and $f''(x) \geq 0$ on the segment $[0, a]$. Then

$$2 \int_{0}^{a} f(x) \left(1 + f'(x)^2\right)^{1/2} dx \leq b\,(a^2 + b^2)^{1/2},$$

with equality if and only if $f(x) = (b/a)\, x$.

Reference

Pólya, G.: Problem 4264. Amer. Math. Monthly **54**, 479 (1947).

3.7.42 Let H be a real function depending on t and $u_1, \ldots, u_n$, defined for $0 \leq t \leq 1$, $-\infty < u_i < +\infty$, for $i = 1, \ldots, n$, and having continuous second derivatives. In order that

$$\int_{0}^{1} H\left(t; f_1(t), \ldots, f_n(t)\right) dt \leq \int_{0}^{1} H\left(t; g_1(t), \ldots, g_n(t)\right) dt$$

holds for each system of decreasing bounded functions f_i, g_i $(1 \leq i \leq n)$ satisfying

$$\int_{0}^{x} f_i(t)\, dt \leq \int_{0}^{x} g_i(t)\, dt \qquad (0 \leq x \leq 1,\ 1 \leq i \leq n)$$

and

$$\int_{0}^{1} f_i(t)\, dt = \int_{0}^{1} g_i(t)\, dt \qquad (1 \leq i \leq n),$$

it is necessary and sufficient that

$$\frac{\partial^2 H}{\partial u_i\, \partial u_j} \geq 0 \qquad (i, j = 1, \ldots, n)$$

and

$$\frac{\partial^2 H}{\partial t\, \partial u_i} \leq 0 \qquad (i = 1, \ldots, n).$$

Reference

Fan, K., and G. G. Lorentz: An integral inequality. Amer. Math. Monthly **61**, 626–631 (1954).

3.7.43 If f'' is continuous on $[0, 1]$ and if $f(0) = f(1) = 0$, then

(1) $$\int_0^1 |f''(x)/f(x)|\, dx > 4.$$

This inequality is the best possible.

Remark. Inequality (1) is due to A. M. LYAPUNOV (see 3.9.66). It is a special case of an inequality due to A. BEURLING, which is, in turn, a special case of an inequality due to G. BORG. See [1]—[3].

Inequality (1) has important applications in ordinary differential equations. See, for example, G. BORG [4]. For similar applications, as well as generalizations of (1) to complex differential equations and univalent functions, see Z. NEHARI [5] and P. R. BEESACK [6].

References

1. BORG, G.: Über die Stabilität gewisser Klassen von linearen Differentialgleichungen. Ark. Mat. Astr. Fys. **31 A**, No. 1 (1944), 31 pp.
2. CORDES, H. O.: An inequality of G. Borg. Amer. Math. Monthly **63**, 27—29 (1956).
3. COHEN, H. J.: Problem E 1205. Amer. Math. Monthly **63**, 582—583 (1956).
4. BORG, G.: On a Liapounoff criterion of stability. Amer. J. Math. **71**, 67—70 (1949).
5. NEHARI, Z.: Univalent functions and linear differential equations. In: Lectures on Functions of a Complex Variable. Ann Arbor 1955, pp. 148—151.
6. BEESACK, P. R.: Nonoscillation and disconjugacy in the complex domain. Trans. Amer. Math. Soc. **81**, 211—242 (1956).

3.7.44 Let f be a convex and a nonnegative function, and let F, L be increasing on $[0, +\infty)$ and absolutely continuous, let $F(0) = 0$ and let $G = F \cdot L$. Then

$$\int_0^{+\infty} G'\left(f(x)\right) dx \leq L(\max f) \int_0^{+\infty} F'\left(f(x)\right) dx.$$

If $F'(0) = 0$, $F'(x) > 0$ for $x > 0$, finite equality holds if and only if, for some $a, b > 0$,

$$f(x) = b(1 - x/a) \text{ for } 0 \leq x \leq a \text{ and } f(x) = 0 \text{ for } x > a.$$

Remark. This result is due to L. SHEPP and is an answer to Problem **4954** proposed by D. J. NEWMAN in Amer. Math. Monthly **69**, 321—322 (1962). In fact, the above result generalizes the following inequality of NEWMAN:

If $f(x) \geq 0$ and if f is a convex function, then

$$\int_0^{+\infty} f(x)^2\, dx \leq \frac{2}{3} (\max f(x)) \int_0^{+\infty} f(x)\, dx.$$

The constant 2/3 is the best possible.

3.7.45 Let the real function f be differentiable on $[a, b]$, and let f' be continuous and monotone. If, for $a \leq t_1 < t_2 \leq b$,

$$\frac{t_2 - t_1}{|f'(t_2) - f'(t_1)|} < K,$$

where K does not depend on t_1 and t_2, then

$$\left| \int_a^b \sin f(t)\, dt \right| < M \sqrt{K}.$$

J. G. van der Corput and E. Landau [1] have found that a value of M is $4\sqrt{2}$. E. Landau [2] showed later that the best possible constant is $M = 2\sqrt{2} \max\limits_{0 \leq x \leq \pi} \int_0^{\sqrt{x}} \sin(x - t^2)\, dt$.

References

1. van der Corput, J. G., and E. Landau: Über Gitterpunkte in ebenen Bereichen. Nachr. Ges. Wiss., Math.-Phys. Kl. 1920, pp. 135–171.
2. Landau, E.: Über eine Integralungleichung. Christiaan Huygens **1**, 235–237 (1921/22).

3.7.46 If a function f possesses a second derivative on (a, b), then, for $f'(b) \neq 0$,

$$\int_a^b f(x)\, dx \gtrless (b - a) f(a) + \frac{1}{2} \frac{(f(b) - f(a))^2}{f'(b)}$$

according as $f''(x) \gtrless 0$ on (a, b).

Reference

Kesava Menon, P.: Some integral inequalities. Math. Student **11**, 36–38 (1943)

3.7.47 Let $f_1, \ldots, f_n$ be convex functions, defined on $0 \leq x \leq 1$, such that

$$f_k(x) \geq 0, \quad f_k(0) = 0 \qquad (k = 1, \ldots, n);$$

then

$$\int_0^1 f_1(x) \cdots f_n(x)\, dx \geq \frac{2^n}{n+1} \left(\int_0^1 f_1(x)\, dx \right) \cdots \left(\int_0^1 f_n(x)\, dx \right).$$

Reference

Andersson, B. J.: An inequality for convex functions. Nord. Mat. Tidskr. **6**, 25–26 (1958).

3.7.48 Consider a closed real interval $[a, b]$ and real numbers α and β with $0 < \alpha < \beta$. Let f be an integrable, concave and monotonely decreasing function on $[a, b]$ for which $f(a) = \beta$ and $f(b) = \alpha$. Then the following inequalities hold [1]:

$$1 \leq \frac{1}{(b-a)^2} \int_a^b f(x)\,dx \int_a^b \frac{1}{f(x)}\,dx \leq \frac{\left(\frac{\beta \log(\beta/\alpha)}{\beta-\alpha} - \frac{\beta+\alpha}{2\beta}\right)^2}{2(\beta-\alpha)\left(\frac{\beta\log(\beta/\alpha)}{\beta-\alpha} - 1\right)}.$$

These inequalities improve a result given in [2].

References

1. WILKINS, J. E.: The average of the reciprocal of a function. Proc. Amer. Math. Soc. **6**, 806–815 (1955).
2. PÓLYA, G., and G. SZEGÖ: Aufgaben und Lehrsätze aus der Analysis, vol. 1. Berlin 1925, p. 57 and p. 214.

3.7.49 Let m and n be nonnegative numbers and $m \leq n$. Then

$$\left((m+1)\int_0^1 f(x)^m dx\right)^{1/m} \leq \left((n+1)\int_0^1 f(x)^n\,dx\right)^{1/n},$$

where $f(x)$ is a convex function in $0 \leq x \leq 1$ and $f(0) = 0$.

Reference

THUNSDORFF, H.: Konvexe Funktionen und Ungleichungen. Inaugural-Dissertation, Göttingen 1932, 40 pp.

3.7.50 Let $x_1, \ldots, x_n$ be real numbers such that $x_1 < \cdots < x_n$, and let

$$f(x; x_1, \ldots, x_n) = \prod_{k=1}^n (x - x_k),$$

$$M = \max_{x_1 < x < x_n} |f(x; x_1, \ldots, x_n)|,$$

$$g(x_1, \ldots, x_n) = \frac{1}{M}\int_{x_1}^{x_n} f(x; x_1, \ldots, x_n)\,dx.$$

Then

$$(-1)^{n+1-k} \frac{\partial g(x_1, \ldots, x_n)}{\partial x_k} > 0.$$

This is a conjecture due to D. Ž. DJOKOVIĆ. See Problem 5311. Amer. Math. Monthly **73**, 788 (1966). No solution has appeared until now.

3.7.51 Let f, g, h be real functions defined for $x \in A$, such that f, g are integrable and h measurable and bounded on A. Let

$$A(y) = \{x \mid h(x) \geq y\}, \qquad B(y) = A - A(y) = \{x \mid h(x) < y\}.$$

If

$$\int_{A(y)} f(x)\,dx \geq \int_{A(y)} g(x)\,dx \quad \text{for all } y \in [0, +\infty)$$

and if

$$\int_{B(y)} f(x)\,dx \leq \int_{B(y)} g(x)\,dx \quad \text{for all } y \in (-\infty, 0),$$

then

$$\int_A g(x)\,h(x)\,dx \leq \int_A f(x)\,h(x)\,dx.$$

Remark. As a special case of this result, due to D. Banks [1], we have an inequality of P. R. Beesack [2], which in turn, as a consequence, yields an inequality of K. Tatarkiewicz [3].

References

1. Banks, D.: An integral inequality. Proc. Amer. Math. Soc. **14**, 823–828 (1963).
2. Beesack, P. R.: A note on an integral inequality. Proc. Amer. Math. Soc. **8**, 875–879 (1957).
3. Tatarkiewicz, K.: Sur une inégalité intégrale. Ann. Univ. Mariae Curie-Skłodowska **A 7**, 83–87 (1953).

3.7.52 Let f be a real-valued, positive, continuous, increasing function defined on $I = [0, 1)$. Then there exist two convex functions g_1 and g_2 on I such that $0 \leq g_1(x) \leq f(x) \leq g_2(x)$, and

$$2\int_0^1 g_1(x)\,dx \geq \int_0^1 f(x)\,dx \geq \frac{1}{2}\int_0^1 g_2(x)\,dx.$$

Constants 2 and 1/2 are the best possible.

The above result is due to A. S. Besicovitch and Roy O. Davies [1]. In connection with this T. Nishiura and F. Schnitzer [2] proved the following:

Let $x = (x_1, \ldots, x_n)$, $h = (h_1, \ldots, h_n)$. Let $f\colon I^n \to R$ be such that $f(x) \geq 0$ and $f(x+h) - f(x) \geq 0$ for $h_i \geq 0$ $(i = 1, \ldots, n)$, and $x + h \in I^n$. Then there exist two convex functions g_1 and g_2 such that $0 \leq g_1(x) \leq f(x) \leq g_2(x)$, and

$$(n+1)!\int g_1(x)\,dx \geq \int f(x)\,dx \geq \frac{n!}{(n+1)^n}\int g_2(x)\,dx.$$

Constants $(n+1)!$ and $n!/(n+1)^n$ are the best possible.

This result of T. Nishiura and F. Schnitzer appears as a corollary of a more general result also formulated and proved in [2].

References

1. Besicovitch, A. S., and R. O. Davies: Two problems on convex functions. Math. Gaz. **49**, 66–69 (1965).
2. Nishiura, T., and F. Schnitzer: Monotone functions and convex functions. Michigan Math. J. **12**, 481–485 (1965).

3.7.53 In the real interval $[a, b]$ let the real-valued function f be continuously four times differentiable. Then

$$\frac{1}{2}(b-a)\big(f(a)+f(b)\big) \leq \int_a^b f(x)\,dx + \frac{1}{12}(b-a)^2\big(f'(b)-f'(a)\big)$$
$$+\frac{1}{384}(b-a)^4\int_a^b \max\big(-f^{(4)}(x), 0\big)\,dx.$$

Reference

ATKINSON, F. V.: Some further estimates concerning sums of powers of complex numbers. Acta Math. Acad. Sci. Hung. **20**, 193–210 (1969).

3.7.54 If $0 \leq a_1 < a_2 < \cdots < a_{2n+1} \leq 1$ and

$$f(x) = \frac{(x-a_2)(x-a_4)\cdots(x-a_{2n})}{(x-a_1)(x-a_3)\cdots(x-a_{2n+1})},$$

then, for $0 < t < 1$,

$$\text{(1)}\qquad \frac{\Gamma\left(\frac{1}{2}+\frac{t}{2}\right)\Gamma\left(1-\frac{t}{2}\right)}{(1-t)\sqrt{\pi}} \leq \int_0^1 |f(x)|^t\,dx \leq \frac{2^t}{1-t}.$$

Equality holds in the first inequality of (1) if and only if $f(x) = \frac{1}{x-1/2}$, and in the second if and only if $f(x) = \frac{x-1/2}{x(x-1)}$.

Reference

HARDY, G. H., and N. LEVINSON: Inequalities satisfied by a certain definite integral. Bull. Amer. Math. Soc. **43**, 709–716 (1937).

3.7.55 Let A be a given negative semidefinite quadratic form

$$A(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \;\; (a_{11}, a_{22} \leq 0;\; a_{11}a_{22} - a_{12}^2 \geq 0).$$

If $D(M)$ is a parallelogram in the xy-plane, whose centre is M, then the function p defined by

$$p(M) = \iint_{D(M)} e^{A(x,y)}\,dx\,dy$$

is a logarithmically concave function of M.

Reference

BODIN, N., and V. ZALGALLER: On the concavity of some functions connected with the two-dimensional normal distribution (Russian). Litovsk. Mat. Sb. **7**, 389–393 (1968).

3.8 Inequalities in the Complex Domain

In several Sections of this book inequalities involving complex numbers or complex functions appear. In this Section several unconnected inequalities which cannot be incorporated in other Sections are cited. However, this Section could be considerably more extensive if, for example, inequalities related to properties of univalent or multivalent functions were included. Because of the lack of space we have restricted ourselves to some particular presented results.

Concerning these topics, see books [1] of P. Montel and [2] of W. K. Hayman, which contain many inequalities with bibliographical references.

References

1. Montel, P.: Leçons sur les fonctions univalentes et multivalentes. Paris 1933.
2. Hayman, W. K.: Multivalent Functions. Cambridge 1958.

3.8.1 If $z_1, \ldots, z_n$ are complex numbers, then

$$1 \leq |1 + z_1| + |z_1 + z_2| + |z_2 + z_3| + |z_3|,$$

$$1 \leq |1 + z_1| + |z_1 + 2z_2| + |2z_2 + 3z_3| + \cdots + |(n-1) z_{n-1} + nz_n| + |nz_n|.$$

Proof. We have

$$1 = |(1 + z_1) - (z_1 + z_2) + (z_2 + z_3) - z_3| \leq |1 + z_1| + |z_1 + z_2| + |z_2 + z_3| + |z_3|,$$

and, similarly,

$$\begin{aligned} 1 &= |(1 + z_1) - (z_1 + 2z_2) + (2z_2 + 3z_3) \\ &\quad - \cdots + (-1)^{n-1} ((n-1) z_{n-1} + nz_n) + (-1)^n nz_n| \\ &\leq |1 + z_1| + |z_1 + 2z_2| + |2z_2 + 3z_3| \\ &\quad + \cdots + |(n-1) z_{n-1} + nz_n| + |nz_n|. \end{aligned}$$

3.8.2 If a and b are two complex numbers with

$$|\arg a - \arg b| \leq \theta \leq \pi,$$

then, for positive integer n,

$$|a - b|^n \leq (|a|^n + |b|^n) \max\left(1, 2^{n-1} \sin^n \frac{\theta}{2}\right).$$

Reference

van der Blij, F.: About generalizations of the triangle inequality. Simon Stevin 25, 231–235 (1946/47).

3.8.3 Let α be a real number and let $0 < \theta < \pi/2$. If $z_1, \ldots, z_n$ are complex numbers such that

$$\alpha - \theta \leq \arg z_\nu \leq \alpha + \theta \quad (\nu = 1, \ldots, n),$$

then

$$\left|\sum_{\nu=1}^{n} z_\nu\right| \geq (\cos\theta) \sum_{\nu=1}^{n} |z_\nu|. \tag{1}$$

Proof. We have

$$\left|\sum_{\nu=1}^{n} z_\nu\right| = \left|e^{-i\alpha} \sum_{\nu=1}^{n} z_\nu\right| \geq \operatorname{Re}\left(e^{-i\alpha} \sum_{\nu=1}^{n} z_\nu\right)$$
$$= \sum_{\nu=1}^{n} |z_\nu| \cos(-\alpha + \arg z_\nu) \geq (\cos\theta) \sum_{\nu=1}^{n} |z_\nu|.$$

Remark. Inequality (1) is a complementary triangle inequality. It is difficult to say where it appeared for the first time in literature. We have found that the special case $\alpha = \theta = \pi/4$ was proved by M. Petrovitch [1] in 1917. The general case of this inequality appears in a later paper [2] of M. Petrovitch. He also applied (1) to derive some inequalities for integrals. Inequality (1) can be found also in J. Karamata's book [3] (pp. 300–301). In [3] one can find the following proposition:

If f is a complex-valued integrable function defined in the interval $a \leq x \leq b$ and

$$-\theta \leq \arg f(x) \leq +\theta \quad (0 < \theta < \pi/2),$$

then

$$\left|\int_a^b f(x)\, dx\right| \geq (\cos\theta) \int_a^b |f(x)|\, dx.$$

J. Karamata has also published inequality (1) in another book [4] (p. 155).

Inequality (1) has been rediscovered by H. S. Wilf [5] in 1963.

Generalization. The generalization of (1) to Hilbert and Banach spaces has been given recently by J. B. Diaz and F. T. Metcalf [6]. We quote their theorems which hold in any Hilbert space H:

Let a be a unit vector in H. Suppose the vectors $x_1, \ldots, x_n$ satisfy

$$0 \leq r \leq \frac{\operatorname{Re}(x_i, a)}{|x_i|} \qquad (i = 1, \ldots, n),$$

whenever $x_i \neq 0$. Then

$$r(|x_1| + \cdots + |x_n|) \leq |x_1 + \cdots + x_n|,$$

where equality holds if and only if

$$x_1 + \cdots + x_n = r(|x_1| + \cdots + |x_n|)\, a.$$

Let $a_1, \ldots, a_m$ be orthonormal vectors in H. Suppose the vectors $x_1, \ldots, x_n$ satisfy

$$0 \leq r_k \leq \frac{\operatorname{Re}(x_i, a_k)}{|x_i|} \qquad (i = 1, \ldots, n,\ k = 1, \ldots, m),$$

whenever $x_i \neq 0$. Then

$$(r_1^2 + \cdots + r_m^2)^{1/2} (|x_1| + \cdots + |x_n|) \leq |x_1 + \cdots + x_n|,$$

where equality holds if and only if

$$x_1 + \cdots + x_n = (|x_1| + \cdots + |x_n|)\,(r_1 a_1 + \cdots + r_m a_m).$$

The following variant of (1) for $\theta = \pi/2$ appears in M. MARDEN [7] (p. 1): If each complex number z_ν ($\nu = 1, \ldots, n$) has the properties that $z_\nu \neq 0$ and

$$\alpha \leq \arg z_\nu < \alpha + \pi,$$

then their sum $z_1 + \cdots + z_n$ cannot vanish.

References

1. PETROVITCH, M.: Module d'une somme. Enseignement Math. 19, 53–56 (1917).
2. PETROVITCH, M.: Théorème sur les intégrales curvilignes. Publ. Math. Univ. Belgrade 2, 45–59 (1933).
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4. KARAMATA, J.: Complex Number with Application to Elementary Geometry (Serbian). Beograd 1950.
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6. DIAZ, J. B., and F. T. METCALF: A complementary triangle inequality in Hilbert and Banach spaces. Proc. Amer. Math. Soc. 17, 88–97 (1966).
7. MARDEN, M.: The geometry of the zeros of a polynomial in a complex variable. Amer. Math. Soc. Math. Surveys, vol. 3. New York 1949.

3.8.4 If z_1 and z_2 are complex numbers, and if c is a positive number, then

$$|z_1 + z_2|^2 \leq (1 + c)\,|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2, \tag{1}$$

with equality if and only if $z_2 = cz_1$.

This inequality is due to H. BOHR [1], p. 78.

Generalization. In the book [2] by J. W. ARCHBOLD the following generalization of (1) can be found:

If $a_1, \ldots, a_n$ are positive numbers such that $\sum_{k=1}^{n} 1/a_k = 1$, then

$$|z_1 + \cdots + z_n|^2 \leq a_1\,|z_1|^2 + \cdots + a_n\,|z_n|^2. \tag{2}$$

Proof of (2). If a_k, b_k ($k = 1, \ldots, n$) are real numbers such that $\sum_{k=1}^{n} 1/a_k = 1$, then by CAUCHY's inequality (see 2.1.2) we have

$$\sum_{k=1}^{n} a_k b_k^2 = \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} a_k b_k^2\right) \geq \left(\sum_{k=1}^{n} b_k\right)^2.$$

Putting $b_k = |z_k| \Big/ \sum_{i=1}^{n} |z_i|$ for $k = 1, \ldots, n$, we get

$$\sum_{k=1}^{n} a_k\,|z_k|^2 - \left(\sum_{k=1}^{n} |z_k|\right)^2 \geq 0. \tag{3}$$

Since

$$\sum_{k=1}^{n} |z_k| \geq \left| \sum_{k=1}^{n} z_k \right|,$$

(3) yields (2).

This is a proof of G. Kalajdžić.

Remark. A. Mąkowski [3] has proved the following inequalities which are related to Bohr's inequality (1) in the case when z_1 and z_2 are real numbers: If a, b, α are real numbers and $c > 0$, then

$$(a - b)^2 \sin \alpha + (a + b)^2 \cos \alpha \leq (1 + c\,|\cos 2\alpha|)\, a^2 + \left(1 + \frac{|\cos 2\alpha|}{c}\right) b^2$$
$$\leq (1 + c)\, a^2 + \left(1 + \frac{1}{c}\right) b^2.$$

References

1. Bohr, H.: Zur Theorie der fastperiodischen Funktionen I. Acta Math. **45**, 29–127 (1924).
2. Archbold, J. W.: Algebra. London 1958, p. 75.
3. Mąkowski, A.: Boletin Mat. 34, 11 (1961).

3.8.5 If $a_1, \ldots, a_n$ are real and $z_1, \ldots, z_n$ complex numbers, then

$$(1) \qquad \left| \sum_{k=1}^{n} a_k z_k \right|^2 \leq \frac{1}{2} \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} |z_k|^2 + \left| \sum_{k=1}^{n} z_k^2 \right| \right),$$

which improves Cauchy's inequality.

Equality holds if and only if, for $k = 1, \ldots, n$, $a_k = \operatorname{Re}(\lambda z_k)$, where λ is a complex number, and $\sum_{k=1}^{n} \lambda^2 z_k^2$ is real and nonnegative.

Proof. By a simultaneous rotation of all the z_k's about the origin, we get $\sum_{k=1}^{n} a_k z_k \geq 0$. This rotation does not affect the moduli

$$\left| \sum_{k=1}^{n} a_k z_k \right|, \qquad \left| \sum_{k=1}^{n} z_k^2 \right|, \qquad |z_k| \quad (k = 1, \ldots, n).$$

Thence, it is sufficient to prove inequality (1) for the case $\sum_{k=1}^{n} a_k z_k \geq 0$.

If we put $z_k = x_k + i y_k$ $(k = 1, \ldots, n)$, then

$$\left| \sum_{k=1}^{n} a_k z_k \right|^2 = \left(\sum_{k=1}^{n} a_k x_k \right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} x_k^2 \right),$$

where we made use of Cauchy's inequality for real numbers (see 2.1.2). Since

$$2x_k^2 = |z_k|^2 + \operatorname{Re} z_k^2,$$

we obtain

$$\left| \sum_{k=1}^{n} a_k z_k \right|^2 \leq \frac{1}{2} \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} |z_k|^2 + \sum_{k=1}^{n} \operatorname{Re} z_k^2 \right).$$

From this inequality we get (1) taking into account that

$$\sum_{k=1}^{n} \operatorname{Re} z_k^2 = \operatorname{Re} \sum_{k=1}^{n} z_k^2 \leq \left| \sum_{k=1}^{n} z_k^2 \right| .$$

Remark. Inequality (1) is sharper than CAUCHY's inequality

$$\left| \sum_{k=1}^{n} a_k z_k \right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2 \right) \left(\sum_{k=1}^{n} |z_k|^2 \right),$$

which also holds if we allow the a_k's to be complex, since

$$\left| \sum_{k=1}^{n} z_k^2 \right| \leq \sum_{k=1}^{n} |z_k|^2 .$$

Reference

DE BRUIJN, N. G.: Problem 12. Wisk. Opgaven **21**, 12—14 (1960).

3.8.6 Let a_{ik} be elements of an $n \times n$ complex matrix A. Then,

$$\operatorname{Re} \left(\sum_{i,k} a_{ik} a_{ki} \right) \leq \sum_{i,k} |a_{ik}|^2 ,$$

with equality if and only if A is hermitian.

If z_k, for $k = 1, \ldots, n$, are complex numbers, then

$$\operatorname{Re} (z_1 z_2 + \cdots + z_{n-1} z_n + z_n z_1) \leq \sum_{k} |z_k|^2 ,$$

equality holding if and only if $z_1 = z_3 = z_5 = \cdots = \overline{z_2} = \overline{z_4} = \overline{z_6} = \cdots$.

Reference

ARGHIRIADE, E.: A characteristic property of hermitian matrices. An. Univ. Timişoara Ser. Şti. Mat.-Fiz. No. **3**, 17—20 (1965).

3.8.7 If $z_1, \ldots, z_n$ $(n \geq 3)$ are complex numbers, and $z_{n+1} = z_1$, then

$$\sum_{k=1}^{n} |z_k - z_{k+1}|^2 \geq 2 \left(\tan \frac{\pi}{n} \right) \operatorname{Im} \left(\sum_{k=1}^{n} \overline{z_k} z_{k+1} \right),$$

with equality if and only if

$$z_k = \alpha \exp \frac{2\pi i k}{n} + \beta \qquad (1 \leq k \leq n),$$

where α and β are arbitrary complex numbers.

Reference

FAN, K., O. TAUSSKY and J. TODD: An algebraic proof of the isoperimetric inequality for polygons. J. Washington Acad. Sci. **45**, 339—342 (1955).

3.8.8 If $a_1, \ldots, a_n$ are complex numbers such that $\sum_{k=1}^{n} |a_k| \leq h < 1$, then

$$\left| \prod_{k=1}^{n} (1 + a_k) - 1 - \sum_{k=1}^{n} a_k \right| \leq \frac{h^2}{1 - h} .$$

If $a_1, \ldots, a_n$ are complex numbers such that

$$\left|\prod_{k\in I}(1+a_k)-1\right| \leq \frac{1}{81}$$

for all subsets I of $\{1, \ldots, n\}$, then

$$\sum_{k=1}^{n}|a_k| \leq \frac{8}{81}.$$

Reference

BOURBAKI, N.: Topologie générale. 2nd ed., Paris 1955, Chap. 5—8, pp. 110—111.

3.8.9 Let z_1 and z_2 be complex numbers, and let u and v be real numbers such that $u \neq 0$, $v \neq 0$ and $u + v \neq 0$. Then

$$\frac{|z_1+z_2|^2}{u+v} \leq \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \quad \text{for} \quad \frac{1}{u}+\frac{1}{v} > 0,$$

and

$$\frac{|z_1+z_2|^2}{u+v} \geq \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \quad \text{for} \quad \frac{1}{u}+\frac{1}{v} < 0.$$

Equality occurs if and only if $vz_1 = uz_2$.

Proof. The above inequalities follow from the following identity:

$$\frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} - \frac{|z_1+z_2|^2}{u+v} = \frac{|vz_1 - uz_2|^2}{uv(u+v)}.$$

Remark. The above inequalities are in connection with 3.8.4.

Reference

BERGSTRÖM, H.: A triangle-inequality for matrices. C.R. 11e Congrès Math. Scandinaves, Trondheim 1949, pp. 264—267 (1952).

3.8.10 Given $n \geq 2$, $\varepsilon > 0$ and complex numbers $z_1, \ldots, z_n$ and $a_1, \ldots, a_n$, then

$$(1) \qquad \sum_{k=1}^{n}|z_k - a_k| < \varepsilon \Rightarrow \left|\prod_{k=1}^{n} z_k - \prod_{k=1}^{n} a_k\right| < \varepsilon \sum_{k=0}^{n-1} S_k \varepsilon^{n-1-k},$$

where S_k denotes the k-th elementary symmetric function of $|a_1|, \ldots, |a_n|$.

Proof. If $|z_1 - a_1| + |z_2 - a_2| < \varepsilon$, then

$$\begin{aligned} |z_1z_2 - a_1a_2| &= |(z_1-a_1)(z_2-a_2) + a_1(z_2-a_2) + a_2(z_1-a_1)| \\ &\leq |z_1-a_1|\,|z_2-a_2| + |a_1|\,|z_2-a_2| + |a_2|\,|z_1-a_1| \\ &< \varepsilon(\varepsilon + |a_1| + |a_2|), \end{aligned}$$

i.e.,

$$(2) \qquad |z_1 - a_1| + |z_2 - a_2| < \varepsilon \Rightarrow |z_1z_2 - a_1a_2| < \varepsilon(\varepsilon + |a_1| + |a_2|).$$

Hence, (1) holds for $n = 2$.

Let us suppose that assertion (1) holds for some fixed $n \geq 2$. We shall prove that it is also true for $n + 1$. We suppose that

$$\sum_{k=1}^{n+1} |z_k - a_k| < \varepsilon.$$

Using the identity

$$\prod_{k=1}^{n+1} z_k - \prod_{k=1}^{n+1} a_k = \left(\prod_{k=1}^{n} z_k - \prod_{k=1}^{n} a_k\right)\left((z_{n+1} - a_{n+1}) + a_{n+1}\right)$$
$$+ \left(\prod_{k=1}^{n} a_k\right)(z_{n+1} - a_{n+1})$$

together with (1) and (2), we obtain

$$\left|\prod_{k=1}^{n+1} z_k - \prod_{k=1}^{n+1} a_k\right| < (\varepsilon + |a_{n+1}|) \sum_{k=0}^{n-1} S_k \varepsilon^{n-k} + \varepsilon\, |a_{n+1}|\, S_n$$
$$= \varepsilon^{n+1} + \varepsilon \sum_{k=1}^{n} (S_k + |a_{n+1}|\, S_{k-1})\, \varepsilon^{n-k}$$
$$= \varepsilon \sum_{k=0}^{n} S_k^* \varepsilon^{n-k}.$$

Here, S_k^* is the k-th elementary symmetric function of $|a_1|, \ldots, |a_n|$, $|a_{n+1}|$.

The proof by induction is complete.

This result is due to D. S. Mitrinović.

3.8.11 If

(1) $$d(a, b) = \frac{|a - b|}{(1 + |a|^2)^{1/2}\, (1 + |b|^2)^{1/2}},$$

then

(2) $$d(a, b) \leq d(a, c) + d(c, b),$$

where a, b, c are complex numbers.

Proof. Starting from

$$(a - b)\,(1 + c\bar{c}) = (a - c)\,(1 + b\bar{c}) + (c - b)\,(1 + a\bar{c}),$$

it follows that

(3) $$|a - b|\,(1 + |c|^2) \leq |a - c|\,|1 + b\bar{c}| + |c - b|\,|1 + a\bar{c}|.$$

Applying the inequality $|\bar{u} - v|^2 \geq 0$, written in its equivalent form

$$(1 + uv)\,(1 + \overline{uv}) \leq (1 + |u|^2)\,(1 + |v|^2),$$

to $|1 + b\bar{c}|$, one finds that

$$(4) \qquad |1 + b\bar{c}|^2 = (1 + b\bar{c})(1 + c\bar{b}) \leq (1 + |b|^2)(1 + |\bar{c}|^2) = (1 + |b|^2)(1 + |c|^2).$$

Since

$$(5) \qquad |1 + \bar{a}c|^2 \leq (1 + |a|^2)(1 + |c|^2),$$

inequality (3) reduces to (2) by the use of (4) and (5).

This elegant proof was given by S. Kakutani (see E. Hille: Analytic Function Theory, vol. 1. Boston-New York 1959, p. 49).

Remark. $d(a, b)$ is called the chordal distance of a and b. It can easily be proved that $d(a, b)$ also satisfies

$$(6) \qquad \begin{aligned} &d(a, a) = 0, \\ &d(a, b) > 0 \quad (a \neq b), \\ &d(a, b) = d(b, a). \end{aligned}$$

Since d, given by (1), satisfies (2), together with (6), it is a metric function on the set of complex numbers.

Function d also satisfies $d(a, b) \leq 1$, which can easily be proved.

3.8.12 If a and b are nonzero complex numbers, then

$$(1) \qquad |a - b| \geq \frac{1}{2}(|a| + |b|)\left|\frac{a}{|a|} - \frac{b}{|b|}\right|.$$

Equality holds if and only if $|a| = |b|$.

Proof. Putting $a = re^{i\theta}$, $b = -\varrho e^{i\varphi}$ with $r, \varrho > 0$, we get

$$\frac{|a - b|}{|a| + |b|} = \left|\frac{r}{r + \varrho} e^{i\theta} + \frac{\varrho}{r + \varrho} e^{i\varphi}\right| = |u|,$$

$$\frac{1}{2}\left|\frac{a}{|a|} - \frac{b}{|b|}\right| = \left|\frac{e^{i\theta} + e^{i\varphi}}{2}\right| = |v|.$$

In the complex plane u lies on the chord, joining the points $e^{i\theta}$ and $e^{i\varphi}$, of the unit circle, and v is precisely the midpoint of this chord. Therefore $|u| \geq |v|$ with equality if and only if $u = v$, i.e., $r = \varrho$. This proves (1).

This proof is due to M. Marjanović.

Remark. A more general result is demonstrated in the paper [1].

Reference

1. Dunkl, C. F., and K. S. Williams: A simple norm inequality. Amer. Math. Monthly **71**, 53–54 (1964).

3.8.13 Let z and t be complex numbers such that $|z| < 1$ and $|t| < 1$. Then

$$\frac{|z| - |t|}{1 - |z|\,|t|} \leq \left|\frac{z - t}{z\bar{t} - 1}\right| \leq \frac{|z| + |t|}{1 + |z|\,|t|} < 1.$$

Reference

BEHNKE, H., and F. SOMMER: Theorie der analytischen Funktionen einer komplexen Veränderlichen. 3rd ed., Berlin-Heidelberg-New York 1965, pp. 336–337.

3.8.14 Let a and c be positive numbers and b a complex number such that

$$f(z) \equiv az\bar{z} + bz + \bar{b}\bar{z} + c \geq 0$$

for every complex number z. Then

$$b\bar{b} \leq ac, \tag{1}$$

$$f(z) \leq (a + c)\,(1 + z\bar{z}), \tag{2}$$

with equality in (1) if and only if $f(z) = 0$ for some value of z.

Proof. Put $z = re^{i\theta}$ and $b = \varrho e^{it}$, where r and ϱ are positive or zero and θ and t are real. Then, by the hypothesis,

$$f(z) \equiv ar^2 + 2\varrho r\cos(\theta + t) + c \geq 0, \tag{3}$$

$$f(z) \geq ar^2 - 2\varrho r + c \geq 0, \tag{4}$$

for all values of r and θ, with equality in the first inequality of (4) for $\cos(\theta + t) = -1$. From (4) it follows that $\varrho^2 - ac \leq 0$, with equality if and only if, for some value $z = z_0$, we have

$$f(z_0) = ar_0^2 - 2\varrho r_0 + c = 0,$$

where $|z_0| = r_0$. Thus (1) is proved.

From $\varrho^2 - ac \leq 0$ it follows that

$$\varrho r \leq r\sqrt{ac} = \sqrt{a \cdot cr^2} \leq \frac{1}{2}(a + cr^2). \tag{5}$$

Now, we deduce from (3) and (5) that

$$f(z) \leq ar^2 + 2\varrho r + c \leq (ar^2 + c) + (a + cr^2)$$
$$= (a + c)\,(r^2 + 1),$$

and (2) is proved.

Remark 1. This result is due to N. ARONSZAJN and can be found in: H. HAMBURGER and M. S. GRIMSHAW: Linear Transformations in n-dimensional Vector Space. Cambridge 1951, pp. 76–77.

Remark 2. If $f(z) \geq 0$ for every z, then the circle

$$az\bar{z} + bz + \bar{b}\bar{z} + c = 0$$

must be imaginary or reduce to a point. This fact also yields (1).

3.8.15 In the following we shall give estimates of the modulus of a homographic function on a circle.

If the point z varies along the circle ($\gamma \neq 0$) or the line ($\gamma = 0$), which is given by

(1) $\gamma z\bar{z} + \bar{\alpha}z + \alpha\bar{z} - \beta = 0 \qquad (\beta, \gamma \text{ real and } \alpha\bar{\alpha} + \beta\gamma > 0)$,

the point

$$w = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

also varies along the circle or the line

(2) $[(\alpha c - \gamma d)\,\bar{d} + (\beta c + \bar{\alpha} d)\,\bar{c}]\, w\bar{w} - [(\alpha c - \gamma d)\,\bar{b} + (\beta c + \bar{\alpha} d)\,\bar{a}]\, w$
$- [(\bar{\alpha}\bar{c} - \gamma\bar{d})\, b + (\beta\bar{c} + \alpha\bar{d})\, a]\,\bar{w} + [(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}] = 0.$

This equation determines a circle if

$$D = (\alpha c - \gamma d)\,\bar{d} + (\beta c + \bar{\alpha} d)\,\bar{c} \neq 0.$$

Comparing the equation $|w - p| = R$ with (2), we get

(3) $$R^2 = p\bar{p} - \frac{(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}}{D}$$

and

(4) $$p = \frac{1}{D}\,[(\bar{\alpha}\bar{c} - \gamma\bar{d})\, b + (\beta\bar{c} + \alpha\bar{d})\, a].$$

These imply that

$$R^2 = \frac{1}{D^2}\{[(\bar{\alpha}\bar{c} - \gamma\bar{d})\, b + (\beta\bar{c} + \alpha\bar{d})\, a]\,[(\alpha c - \gamma d)\,\bar{b} + (\beta c + \bar{\alpha} d)\,\bar{a}]$$
$$- [(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}]\,[(\alpha c - \gamma d)\,\bar{d} + (\beta c + \bar{\alpha} d)\,\bar{c}]\}$$
$$= \frac{1}{D^2}(ad - bc)(\bar{a}\bar{d} - \bar{b}\bar{c})(\alpha\bar{\alpha} + \beta\gamma),$$

and consequently

(5) $$R = \frac{1}{|D|}\,|ad - bc|\,\sqrt{\alpha\bar{\alpha} + \beta\gamma}\,.$$

Starting with (2), we find directly

$$R^2 - |p|^2 = -\frac{1}{D}\,[(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}].$$

So, it follows that

(6) $$\big|R - |p|\big| = \frac{|R^2 - |p|^2|}{R + |p|}$$
$$= \frac{|(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}|}{|ad - bc|\sqrt{\bar{\alpha}\alpha + \beta\gamma} + |(\bar{\alpha}\bar{c} - \gamma\bar{d})\, b + (\beta\bar{c} + \alpha\bar{d})\, a|}.$$

If the point z varies along (1) and if $D \neq 0$ and $ad - bc \neq 0$, starting with

$$||w - p| - |p|| \leq |w| \leq |w - p| + |p|,$$

we get

(7) $$|R - |p|| \leq \left|\frac{az + b}{cz + d}\right| \leq R + |p|,$$

where $|R - |p||$, R and p are given by (6), (5), (4) respectively.

If $D = 0$ and $ad - bc \neq 0$, starting with

$$|A\bar{w} + \bar{A}w| \leq |A\bar{w}| + |\bar{A}w| = 2\,|A|\,|w|,$$

we get, according to (2),

(8) $$\frac{1}{2}\left|\frac{B}{A}\right| \leq \left|\frac{az + b}{cz + d}\right| \qquad (A \neq 0),$$

where

$$A = (\bar{\alpha}\bar{c} - \gamma\bar{d})\,b + (\beta\bar{c} + \alpha\bar{d})\,a,$$

$$B = -[(\alpha a - \gamma b)\,\bar{b} + (\beta a + \bar{\alpha} b)\,\bar{a}].$$

Inequalities (7) and (8) present the requested bounds.

Remark. If z is a real variable, then (7) becomes

$$\frac{|a\bar{b} - b\bar{a}|}{|ad - bc| + |a\bar{d} - b\bar{c}|} \leq \left|\frac{az + b}{cz + d}\right| \leq \frac{|ad - bc| + |a\bar{d} - b\bar{c}|}{|c\bar{d} - d\bar{c}|}.$$

References

MITRINOVIĆ, D. S.: Limitations en module d'une fonction homographique sur un cercle. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **143—155**, 3—4 (1965).

MITRINOVIĆ, D. S.: Problem E 1841. Amer. Math. Monthly **74**, 442—443 (1967).

3.8.16 If n is a natural number and if z and a are complex numbers such that $|z| \leq r$ and $|a| \leq r$, then

$$\left|\frac{z^n - a^n}{z - a} - na^{n-1}\right| \leq \frac{1}{2}\,n\,(n - 1)\,r^{n-2}\,|z - a| \qquad (z \neq a).$$

Proof. We have

$$\begin{aligned}\frac{z^n - a^n}{z - a} - na^{n-1} &= \sum_{k=0}^{n-1} z^k a^{n-1-k} - na^{n-1}\\ &= \sum_{k=0}^{n-1} a^{n-1-k}\,(z^k - a^k)\\ &= \sum_{k=0}^{n-1} (z - a)\,a^{n-1-k}\,(a^{k-1} + za^{k-2} + \cdots + z^{k-1}).\end{aligned}$$

Therefore,

$$\left|\frac{z^n - a^n}{z - a} - na^{n-1}\right|$$

$$\leq |z - a| \sum_{k=0}^{n-1} |a|^{n-1-k} \left(|a|^{k-1} + |z|\,|a|^{k-2} + \cdots + |z|^{k-1}\right)$$

$$\leq |z - a| \sum_{k=0}^{n-1} k r^{n-2}$$

$$= \frac{1}{2} n(n-1) r^{n-2} |z - a|,$$

as asserted.

Remark. Revised from a proof by I. LAZAREVIĆ.

3.8.17 If $\operatorname{Re} z \geq 1$, then for any positive integer n,

(1) $$|z^{n+1} - 1| > |z|^n |z - 1|.$$

Remark. This inequality was conjectured by R. SPIRA in Amer. Math. Monthly **68**, 577 (1961), Problem 4975. A proof of (1) was given in the same journal, **69**, 927–928 (1962). Another proof, given by D. Ž. DJOKOVIĆ, was published in: MITRINOVIĆ **1**, pp. 140–141.

3.8.18 Let $b \neq 0$ be a complex number and $a > 0$. If z is a complex variable satisfying

(1) $$\left|z + \frac{b}{z}\right| = a,$$

then

$$\max_{z \in C} |z| = \frac{1}{2}\left(\sqrt{a^2 + 4|b|} + a\right),$$

$$\min_{z \in C} |z| = \frac{1}{2}\left(\sqrt{a^2 + 4|b|} - a\right),$$

where C is the curve defined by (1).

Remark. For a proof of the above result see: MITRINOVIĆ **2**, pp. 216–218.

3.8.19 Let $0 \leq \arg z \leq \pi$, $-\infty < t < +\infty$, $0 \leq \arg(z + t) \leq \pi$; let n be an integer and $0 \leq n < r \leq n + 1$. Then, uniformly with respect to z,

$$\left|(z + t)^r - \sum_{k=0}^{n} \binom{r}{k} t^k z^{r-k}\right| \leq A_r |t|^r,$$

where the best possible value of A_r is

$$A_r = 1 \qquad \left(0 < r - n \leq \frac{1}{2} \quad \text{or} \quad r = n + 1\right),$$

$$A_r \leq 2^{n+1-r} \sin \frac{\pi(r-n)}{2} \qquad \left(\frac{1}{2} < r - n < 1\right).$$

Reference

KOBER, H.: Approximation by integral functions in the complex domain. Trans. Amer. Math. Soc. **56**, 22 (1944).

3.8.20 If $z = e^{i\theta} \cos \theta$ with $0 < \theta < \pi/2$ and if n is a positive integer, then

$$|1 - z| < |1 - z^n|.$$

Reference

BREUSCH, R.: Problem 5394. Amer. Math. Monthly **75**, 85 (1968).

3.8.21 If z and w are complex numbers and $\lambda \geq 2$, then

$$(1) \qquad 2(|z|^\lambda + |w|^\lambda) \leq |z + w|^\lambda + |z - w|^\lambda.$$

Proof. Without loss of generality, we can assume that $|z| \geq |w|$, so that $w = zre^{i\varphi}$ $(0 \leq r \leq 1; \varphi$ real$)$.

Now, (1) can be written as

$$(2) \qquad 2(1 + r^\lambda) \leq |1 + re^{i\varphi}|^\lambda + |1 - re^{i\varphi}|^\lambda = f(r, \varphi, \lambda),$$

where

$$f(r, \varphi, \lambda) = (1 + r^2 + 2r\cos\varphi)^{\lambda/2} + (1 + r^2 - 2r\cos\varphi)^{\lambda/2}.$$

But

$$\min_\varphi f(r, \varphi, \lambda) = 2(1 + r^2)^{\lambda/2},$$

so that inequality (1) holds if

$$(3) \qquad 1 + r^\lambda \leq (1 + r^2)^{\lambda/2}.$$

The last inequality is true if

$$1 + \frac{\lambda}{2} r^2 \leq (1 + r^2)^{\lambda/2} \quad \text{and} \quad 1 + r^\lambda \leq 1 + \frac{\lambda}{2} r^2.$$

Since these two inequalities hold for $0 \leq r \leq 1$ and $\lambda \geq 2$, inequality (1) is proved.

Reference

VAN DER CORPUT, J. G., and E. W. BETH: Problem 172. Wisk. Opgaven **16**, 421–422 (1937).

3.8.22 Let z be a complex number and $0 < |z| < 1$. Then

$$\frac{1}{4}|z| < |e^z - 1| < \frac{7}{4}|z|.$$

Reference

DIENES, P.: The Taylor Series. Reprint, New York 1957, p. 135.

3.8.23 For all complex numbers $z = x + iy$,

$$|e^z - 1| \leq e^{|z|} - 1 \leq |z|\, e^{|z|}.$$

$$|\sinh y| \leq |\sin z| \leq \cosh y,$$

$$|\sinh y| \leq |\cos z| \leq \cosh y,$$

$$|\operatorname{cosec} z| \leq \operatorname{cosech} |y|,$$

$$|\cos z| \leq \cosh |z|, \qquad |\sin z| \leq \sinh |z|,$$

If $|z| < 1$, then

$$|\cos z| < 2 \quad \text{and} \quad |\sin z| \leq \frac{6}{5}|z|.$$

Reference

ABRAMOWICZ, M., and I. A. STEGUN: Handbook of Mathematical Functions. New York 1965, p. 70 and p. 75.

3.8.24 If $\frac{1}{2} < r \leq 1$ and $|z| \leq \frac{2r - 1}{4r^2}$, then

$$\left|e^z - \frac{4r^2}{4r - 1}\right| < \frac{2r(2r - 1)}{4r - 1}.$$

Reference

WALL, H. S.: Problem 4052. Amer. Math. Monthly **49**, 549 (1942).

3.8.25 For any complex number z and for any natural number n,

$$\left|e^z - \left(1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!}\right)\right| \leq \frac{|z|^{n+1}}{(n + 1)!}\, e^{|z|}.$$

Proof. We have

$$\left|e^z - \left(1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!}\right)\right| = \left|\sum_{k=n+1}^{+\infty} \frac{z^k}{k!}\right| \leq \sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!}$$

$$\leq \frac{|z|^{n+1}}{(n + 1)!} \sum_{k=0}^{+\infty} \frac{|z|^k}{k!} = \frac{|z|^{n+1}}{(n + 1)!}\, e^{|z|}.$$

Reference

GARNIR, H. G.: Fonctions de variables réelles, vol. 1. Louvain-Paris 1963, pp. 369–370.

Remark by P. R. BEESACK. By using a geometric series to majorize $\sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!}$ one gets

$$\left| e^z - \left(1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!}\right) \right| \leq \frac{(n+2)\,|z|^{n+1}}{(n+1)!\,(n+2-|z|)} \quad \text{for } |z| < n+2.$$

This is an improvement of the above at least for all $|z| < n + 1$.

3.8.26 If n is a natural number and z any complex number, then

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| < \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right| < e^{|z|} \frac{|z|^2}{2n}. \tag{1}$$

Proof. For $n = 1$ the above inequalities are true. Suppose that $n \geq 2$. Then

$$\left(1 + \frac{z}{n}\right)^n = 1 + z + \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{z^k}{k!}$$

and

$$\begin{aligned} \left| e^z - \left(1 + \frac{z}{n}\right)^n \right| &= \left| \sum_{k=2}^{+\infty} \frac{z^k}{k!} - \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{z^k}{k!} \right| \\ &= \left| \sum_{k=n+1}^{+\infty} \frac{z^k}{k!} + \sum_{k=2}^{n} \left[1 - \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{z^k}{k!}\right] \right|. \end{aligned}$$

Since all the coefficients are positive, we may replace z by $|z|$ to get the first inequality in (1).

If $k \leq n$, then

$$\frac{1}{k!}\left[1 - \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right] < \frac{1}{k!}\left[1 - \left(1 - \frac{k(k-1)}{2n}\right)\right] \leq \frac{1}{n(k-2)},$$

because

$$(1 - a_1) \cdots (1 - a_n) \geq 1 - \sum_{i=1}^{n} a_i \quad \text{for} \quad a_1, \ldots, a_n \in [0, 1].$$

For $k > n$, we have

$$\frac{1}{k!} \leq \frac{1}{2n(k-2)!}.$$

This yields

$$\left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right| < \frac{1}{2n} \sum_{k=2}^{+\infty} \frac{|z|^k}{(k-2)!} = \frac{|z|^2}{2n} e^{|z|},$$

which, in turn, proves the second inequality in (1).

Remark 1. The inequality

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| < e^{|z|} \frac{|z|^2}{2n}$$

is due to H. D. KLOOSTERMAN. See: Problem 23. Wisk. Opgaven **18**, 70–72 (1943).

Remark 2. Concerning the first inequality in (1) the following result is valid: If

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k \quad \text{with} \quad a_k \geq 0 \quad \text{for} \quad |z| < R,$$

then

$$|f(z)| \leq f(|z|) \quad \text{for} \quad |z| < R,$$

because

$$\left|\sum_{k=0}^{+\infty} a_k z^k\right| \leq \sum_{k=0}^{+\infty} a_k |z|^k \quad \text{for} \quad |z| < R.$$

This statement was also applied in **3.8.25**.

Reference

GARNIR, H. G.: Fonctions de variables réelles, vol. 1. Louvain-Paris **1963**, p. 370.

3.8.27 For the function

$$E(z, p) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

the following inequalities hold

1° $|E(z, p) - 1| \leq |z|^{p+1} \quad (|z| \leq 1)$,

2° $\log |E(z, p)| \leq \begin{cases} |z|^{p+1} & (|z| \leq 1) \\ (2 + \log p)\, |z|^p & (|z| > 1). \end{cases}$

Proof of 1°. For $E(z, p)$ the following expansion is valid:

$$E(z, p) = 1 + \sum_{k=1}^{+\infty} A_{kp} z^k. \tag{1}$$

For the derivative of $E(z, p)$, we have

$$\frac{d}{dz} E(z, p) = -z^p \left(1 + \sum_{k=1}^{+\infty} B_{kp} z^k\right), \tag{2}$$

where B_{kp} $(k = 1, 2, \ldots)$ are positive.

Starting with (1), we get

$$\frac{d}{dz} E(z, p) = \sum_{k=1}^{+\infty} k A_{kp} z^{k-1}. \tag{3}$$

By (2) and (3) one finds that

$$A_{1p} = A_{2p} = \cdots = A_{pp} = 0 \quad \text{and} \quad A_{kp} < 0 \quad (k > p). \tag{4}$$

Using (4), by (1) it follows that

$$\begin{aligned} |E(z, p) - 1| &= \left|\sum_{k=p+1}^{+\infty} A_{kp} z^k\right| \leq \sum_{k=p+1}^{+\infty} |A_{kp}|\, |z^k| \\ &= |z|^{p+1} \sum_{k=p+1}^{+\infty} |A_{kp}|\, |z|^{k-p-1}. \end{aligned} \tag{5}$$

By (5), since $|z| \leq 1$,

$$(6) \qquad |E(z, p) - 1| \leq |z|^{p+1} \sum_{k=p+1}^{+\infty} |A_{kp}|.$$

Since $E(1, p) = 0$, by (1) and (4) follows

$$(7) \qquad \sum_{k=p+1}^{+\infty} A_{kp} = -1.$$

As all the A_{kp}'s are negative for $k > p$, (7) gives

$$(8) \qquad \sum_{k=p+1}^{+\infty} |A_{kp}| = 1.$$

Starting with (6) and using (8), we get

$$|E(z, p) - 1| \leq |z|^{p+1},$$

where $|z| \leq 1$ and p a natural number.

This inequality is due to L. Fejér (see for example [1]).

For 2° see [2].

References

1. Hille, E.: Analytic Function Theory, vol. 1. Boston-New York 1959, p. 227.
2. Hille, E.: Analytic Function Theory, vol. 2. Boston-New York 1962, p. 195.

3.8.28 For $|z| < 1$,

$$|\log(1 + z)| \leq -\log(1 - |z|).$$

Reference

Abramowicz, M., and I. A. Stegun: Handbook of Mathematical Functions. New York 1965, p. 68.

3.8.29 If T_n denotes the sum of the first $n + 1$ terms of the exponential series, then

$$\left| \frac{e^z - T_n}{\dfrac{z^{n+1}}{(n+1)!}} - \frac{(n+2)^2}{n(n+4)} \right| \leq \frac{2(n+2)}{n(n+4)}$$

for $|z| \leq 1$ and $n = 1, 2, \ldots$

Reference

Wall, H. S.: A class of functions bounded in the unit circle. Duke Math. J. 7, 146–153 (1940).

3.8.30 If $0 < a < 1$ and

$$(1) \qquad b = \frac{1}{a}\left(\frac{e^{-a}}{1-a} - 1\right),$$

then

$$(2) \qquad |e^z| \leq |1 + z| \, (1 + b\,|z|),$$

for all complex numbers z satisfying $|z| \leq a$.

Proof. Put $z = x + iy$, $|z| = \varrho \in (0, 1)$. We have

$$\left|\frac{e^z}{1+z}\right|^2 = \frac{e^{2x}}{1 + \varrho^2 + 2x} = f(x) \qquad (-\varrho < x < \varrho).$$

Since $f'(x) < 0$, $(2x < -\varrho^2)$, $f'(x) > 0$ $(2x > -\varrho^2)$ and

$$\frac{e^{-\varrho}}{1-\varrho} > \frac{e^{\varrho}}{1+\varrho} \qquad (0 < \varrho < 1),$$

we conclude that

$$(3) \qquad \max_{|z|=\varrho} \left|\frac{e^z}{1+z}\right| = \frac{e^{-\varrho}}{1-\varrho} = g(\varrho) \qquad (0 < \varrho < 1).$$

From the fact that $g(\varrho)$ is convex on $(0, 1)$, and $g(0) = 1$, $g'(0) = 0$, we deduce that

$$(4) \qquad g(\varrho) \leq 1 + b\varrho \qquad (0 \leq \varrho \leq a),$$

where b is given by (1). Inequality (2) is a consequence of (3) and (4).

Remark. As a consequence of (3) we see that the majorization of $\left|\dfrac{e^z}{1+z}\right|$ in a closed disk $|z| \leq a$ $(0 < a < 1)$ by a function $h(|z|)$ is equivalent to the majorization of $\dfrac{e^{-|z|}}{1-|z|}$.

The above proof is due to S. B. Prešić and D. D. Adamović.

Reference

Mitrinović, D. S.: Problem 85. Mat. Vesnik **5 (20)**, 117—119 (1968).

3.8.31 Let z be a complex number and $|z| < 1$. Then

$$(1) \qquad \frac{|z|}{1+|z|} \leq |\log(1+z)| \leq |z| \frac{1+|z|}{|1+z|}.$$

Proof. To prove the right-hand inequality, we start from the Taylor series for $\log(1 + z)$, which represents the principal value of the logarithm for $|z| < 1$. We have

$$\begin{aligned}(1+z)\log(1+z) &= (1+z)\left(z - \frac{z^2}{2} + \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots\right)\\ &= z + \left(\frac{z^2}{2} - \frac{z^3}{2\cdot 3} + \cdots + (-1)^n \frac{z^n}{(n-1)\,n} + \cdots\right)\\ &= z\left[1 + z\left(\frac{1}{2} - \frac{z}{2\cdot 3} + \cdots + (-1)^n \frac{z^n}{(n+1)(n+2)} + \cdots\right)\right].\end{aligned}$$

Since $|z| < 1$, we get

$$|1+z|\,|\log(1+z)| \leq |z|\left[1 + |z|\left(\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots\right)\right]$$
$$= |z|\left\{1 + |z|\left[\frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots\right]\right\}$$
$$= |z|\,(1 + |z|),$$

whence we have the second inequality of (1).

For the left-hand inequality, after setting $z = |z|\, e^{i\theta} = re^{i\theta}$, we obtain

$$\log(1+z) = \int_0^r \frac{e^{i\theta}}{1 + te^{i\theta}}\, dt$$

and consequently

$$|\log(1+z)| = \left|\int_0^r \frac{1}{1+te^{i\theta}}\, dt\right| \geq \left|\operatorname{Re} \int_0^r \frac{1}{1+te^{i\theta}}\, dt\right|.$$

Since, for $0 \leq t \leq r \leq 1$,

$$\operatorname{Re} \frac{1}{1+te^{i\theta}} = \frac{1 + t\cos\theta}{1 + 2t\cos\theta + t^2} \geq \frac{1}{1+t} \geq \frac{1}{1+r},$$

we conclude that

$$|\log(1+z)| \geq \left|\operatorname{Re} \int_0^r \frac{1}{1+te^{i\theta}}\, dt\right| \geq \int_0^r \frac{1}{1+r}\, d\theta = \frac{r}{1+r},$$

i.e., the first inequality of (1).

The above proof is due to S. E. Warschawski.

Reference

Wall, H. S.: Problem 3965. Amer. Math. Monthly **49**, 72–75 (1942).

3.8.32 Let n be a natural number greater than 1 and let

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}\,[-\log z + \psi(n)] - \sum_{\substack{m=0\\ m\neq n-1}}^{+\infty} \frac{(-z)^m}{(m-n+1)\,m!},$$

where $|\arg z| < \pi$, $\psi(1) = -\gamma$, $\psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m}$, and $\gamma = 0.5772156649\ldots$ is Euler's constant.

We also define

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^{+\infty} \frac{(-1)^n z^n}{n\,n!},$$

where $|\arg z| < \pi$.

For $x > 0$ and $n = 1, 2, \ldots$ the following inequalities hold:

$$\frac{n-1}{n} E_n(x) < E_{n+1}(x) < E_n(x),$$

$$E_n(x)^2 < E_{n-1}(x)\, E_{n+1}(x),$$

$$\frac{1}{x+n} < e^x E_n(x) \leq \frac{1}{x+n-1},$$

$$\frac{d}{dx}\left(\frac{E_n(x)}{E_{n-1}(x)}\right) > 0.$$

If $x > 0$, then

$$\frac{1}{2} \log\left(1 + \frac{2}{x}\right) < e^x E_1(x) < \log\left(1 + \frac{1}{x}\right).$$

Reference

ABRAMOWICZ, M., and I. A. STEGUN: Handbook of Mathematical Functions. New York 1965, p. 229.

3.8.33 If a and c are positive numbers such that $c - a > 1$, and

$$F(a, 1, c; -z) = 1 - \frac{a}{c} z + \frac{a(a+1)}{c(c+1)} z^2 - \cdots, \tag{1}$$

then

$$\left| F(a, 1, c; -z) - \frac{H^2}{2H-1} \right| \leq \frac{H(H-1)}{2H-1} \quad \text{for} \quad |z| \leq 1, \tag{2}$$

where

$$\begin{aligned} H &= F(a, 1, c; 1) \\ &= 1 + \sum_{n=0}^{+\infty} \frac{a(a+1)\cdots(a+n)}{c(c+1)\cdots(c+n)} \\ &= 1 + \sum_{n=0}^{+\infty} \frac{a(a+1)\cdots(a+n)\, n!\,(c+2n+1)}{c(c+1)\cdots(c+n)\,(c-a)\,(c-a+1)\cdots(c-a+n)}. \end{aligned}$$

If $z = -1$, then equality holds in (2); and if $|z| \leq 1$, the real part of $F(a, 1, c; -z)$ is not less than $H/(2H-1)$.

Consider now the hypergeometric function $F(a, 1, c; -z)$ defined by (1), where a and c are real numbers such that $c > a > 0$. Then

$$\left| F(a, 1, c+1; -z) - \frac{c^2}{c^2-a^2} \right| \leq \frac{ac}{c^2-a^2} \quad \text{for} \quad |z| \leq 1; \tag{3}$$

$$\left| F(a, 1, c+1; -z) - \frac{c}{2c-a} \right| \leq \frac{c}{2c-a} \quad \text{for} \quad \operatorname{Re} z \geq -\frac{1}{2}; \tag{4}$$

$$\left| \frac{1}{F(a, 1, c; -z)} - \frac{2c-a+cz}{2c-a} \right| \leq \frac{c-a}{2c-a} |z| \quad \text{for} \quad |z| \leq 1; \tag{5}$$

$$\left| \frac{1}{F(a, 1, c; -z)} - \frac{c+a+az}{c+a} \right| \leq \frac{a}{c+a} |z| \quad \text{for} \quad \operatorname{Re} z \geq -\frac{1}{2}. \tag{6}$$

Remark. From (3) and (4), for $|z| \leq 1$, follow inequalities 3.8.31.

References

WALL, H. S.: A class of functions bounded in the unit circle. Duke Math. J. **7**, 146–153 (1940).
WALL, H. S.: Problem 3965. Amer. Math. Monthly **47**, 491 (1940) and **49**, 72–75 (1942).

3.8.34 Let $z_1, \ldots, z_n$ be arbitrary complex numbers. Then there exists a subset M of $\{1, \ldots, n\}$ such that

$$(1) \qquad \left|\sum_{k\in M} z_k\right| \geq \frac{1}{4\sqrt{2}} \sum_{k=1}^{n} |z_k|.$$

Proof. For $r = 1, 2, 3, 4$ let

$$M_r = \left\{k \mid 1 \leq k \leq n,\ (r-1)\frac{\pi}{2} - \frac{\pi}{4} < \arg z_k \leq (r-1)\frac{\pi}{2} + \frac{\pi}{4}\right\}.$$

By a known inequality we have

$$\left|\sum_{k\in M_r} z_k\right| \geq \frac{1}{\sqrt{2}} \sum_{k\in M_r} |z_k| \qquad (r = 1, 2, 3, 4).$$

Taking M to be that of the M_r's for which the sum $\sum\limits_{k\in M_r} |z_k|$ is maximal, we get (1).

Remark. For a stronger version of (1), see 3.8.36.

Reference

RUDIN, W.: Real and Complex Analysis. New York-St. Louis-San Francisco-Toronto-London-Sydney 1966, p. 119.

3.8.35 If $\alpha_k \in E^n$ $(k = 1, \ldots, m)$, then there exists a subset I of $\{1, \ldots, m\}$ such that

$$(1) \qquad \left|\sum_{k\in I} \alpha_k\right| \geq \frac{1}{2n} \sum_{k=1}^{m} |\alpha_k|.$$

We can prove a stronger inequality

$$(2) \qquad \left|\sum_{k\in I} \alpha_k\right| \geq \frac{1}{2nK_n} \sum_{k=1}^{m} |\alpha_k|,$$

where

$$K_n = \int_0^{\pi/2} \cos^n \theta \, d\theta,$$

i.e.,

$$K_{2r} = \frac{(2r-1)!!}{(2r)!!} \frac{\pi}{2}, \qquad K_{2r+1} = \frac{(2r)!!}{(2r+1)!!}.$$

Proof. For a unit vector α define

$$f(\alpha) = \sum_{k=1}^{m} (\alpha\alpha_k)^+,$$

where, for real x, we write $x^+ = \max(x, 0)$.

Integrating over the unit sphere $|\alpha| = 1$ in E^n, we get

$$\frac{1}{S_n(1)} \int f(\alpha)\, d\alpha = \frac{1}{S_n(1)} \sum_{k=1}^{m} \int (\alpha\alpha_k)^+\, d\alpha,$$

where $S_n(1)$ is the surface area of the unit sphere.

Since the surface area of $S_n(r)$ is

$$S_n(r) = n \cdot 2^n \cdot K_1 \cdots K_n r^{n-1},$$

we get

$$\frac{1}{S_n(1)} \int f(\alpha)\, d\alpha = \frac{1}{S_n(1)} \sum_{k=1}^{m} |\alpha_k| \int_0^{\pi/2} (n-1)\, 2^{n-1} K_1 \cdots K_{n-1} \sin^{n-2}\theta \cos\theta\, d\theta$$

$$= \frac{1}{2nK_n} \sum_{k=1}^{m} |\alpha_k|.$$

Hence, there is α such that

$$f(\alpha) \geq \frac{1}{2nK_n} \sum_{k=1}^{m} |\alpha_k|. \tag{3}$$

Let us define $I = \{k \mid 1 \leq k \leq m;\ \alpha\alpha_k > 0\}$. Then

$$\left|\sum_{k\in I} \alpha_k\right| \geq \sum_{k\in I} \alpha\alpha_k = \sum_{k=1}^{m} (\alpha\alpha_k)^+ = f(\alpha),$$

so that (3) implies (2).

The constant $1/(2nK_n)$ in (2) also is the best possible.

Remark. Inequality (1) appears in: N. Bourbaki: Topologie générale. 2nd ed., Paris 1955, Chap. 5–8, p. 85.

Inequality (2) and its proof are due to D. Ž. Djoković. This result, as well as that in 3.8.36 below, was recently rediscovered by W. W. Bledsoe in Amer. Math. Monthly **77**, 180–182 (1970).

3.8.36 If $z_1, \ldots, z_m$ are complex numbers, then there exists a subset I of $\{1, \ldots, m\}$ such that

$$\left|\sum_{k\in I} z_k\right| \geq \frac{1}{\pi} \sum_{k=1}^{m} |z_k|. \tag{1}$$

Proof. For real x we shall write $x^+ = \max(x, 0)$.

Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$, $r_k = |z_k|$, $0 \leq \theta_k < 2\pi$. We define

$$f(\theta) = \sum_{k=1}^{m} r_k (\cos(\theta - \theta_k))^+.$$

Integrating, we get

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\sum_{k=1}^{m} r_k \int_0^{2\pi} \big(\cos(\theta-\theta_k)\big)^+\,d\theta$$
$$= \frac{1}{2\pi}\sum_{k=1}^{m} r_k \int_{-\pi/2}^{+\pi/2} \cos\theta\,d\theta$$
$$= \frac{1}{\pi}\sum_{k=1}^{m} r_k\,,$$

Consequently, there is a number θ such that

$$f(\theta) \geq \frac{1}{\pi}\sum_{k=1}^{m} |z_k|. \tag{2}$$

If $I = \{k \mid 1 \leq k \leq m,\ \cos(\theta - \theta_k) > 0\}$, then

$$\left|\sum_{k\in I} z_k\right| = \left|e^{-i\theta}\sum_{k\in I} z_k\right|$$
$$\geq \operatorname{Re}\sum_{k\in I} e^{-i\theta} z_k$$
$$= \sum_{k\in I} r_k \cos(\theta-\theta_k)$$
$$= \sum_{k=1}^{m} r_k\big(\cos(\theta-\theta_k)\big)^+$$
$$= f(\theta)\,,$$

so that (2) implies (1).

Remark 1. If at least one $z_k \neq 0$, it can be shown that we have strict inequality in (1). The constant $1/\pi$ is the best possible. This can be proved by taking

$$z_{k+1} = \exp\frac{2k\pi i}{m} \quad (k = 0, 1, \ldots, m-1).$$

Remark 2. Inequality (1) is taken from: N. BOURBAKI: Topologie générale. 2nd ed., Paris 1955, Chap. 5–8, p. 113. The above proof is also indicated therein. Inequality (1) is just the special case $n = 2$ of inequality (2) of 3.8.35.

3.8.37 Let $D_k = \max\limits_{\{z_1,\ldots,z_k\}} \prod\limits_{m=1}^{k} \prod\limits_{\substack{n=1\\ m\neq n}}^{k} |z_m - z_n|$ for $k = 2, 3, \ldots,$ where $\{z_1, \ldots, z_k\}$ ranges over all the sets of k complex numbers which satisfy $|z_m - z_n| \leq 2$ for $m, n = 1, \ldots, k$. Then

$$k^{-k}D_k \geq \left(\sec\frac{\pi}{2k}\right)^{k(k-} \quad > \exp\left(\frac{1}{8}\pi^2\left(1-\frac{1}{k}\right)\right) \quad \text{for} \quad k \equiv 1 \pmod 2;$$

$$k^{-k}D_k > 1 + \frac{1}{32}\frac{\pi^4}{k}\left(1 - \frac{5}{2k} - \frac{2}{k^2}\right) \quad \text{for} \quad k \equiv 2 \pmod 4 \ \text{and} \ k \geq 6;$$

$$k^{-k}D_k > 1 + \frac{1}{32}\frac{\pi^4}{k}\left(1 - \frac{4}{k} - \frac{6}{k^2}\right) \quad \text{for} \quad k \equiv 0 \pmod 4 \ \text{and} \ k \geq 8.$$

We also have $D_2 = 4$, $D_3 = 64$, $D_4 = 256(1 + (2 - \sqrt{3})^2)^2$.
For sufficiently large k,

$$D_k < k^k \exp(15k^{6/7}).$$

In particular, for $k \geq 360$ we have

$$D_k < k^k \exp\left(3k(k-2)^{-1/7}\right).$$

Reference

DANZER, L., and CH. POMMERENKE: Über die Diskriminante von Mengen gegebenen Durchmessers. Monatsh. Math. **71**, 100–113 (1967).

3.8.38 Let f be a regular and univalent function in the disk $|z| < 1$ such that $f(0) = 0$. Then, for $|z_1| < 1$ and $|z_2| < 1$,

$$\left|\frac{f(z_1)}{f(z_2)}\right| \leq \left|\frac{z_1}{z_2}\right| \left(\frac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{1 - |z_1|^2}\right)^2 \cdot \left(\frac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{|1 - z_1\bar{z}_2| - |z_1 - z_2|}\right)^{\frac{|z_2|}{1 - |z_1 z_2|}}.$$

Reference

KRZYŻ, J.: On a problem of P. Montel. Ann. Polon. Math. **12**, 55–60 (1962).

3.8.39 Let f be regular in the disk $|z| \leq 1$. Then, for $|z| < 1$,

$$\frac{(1 - |z|^2)^2}{|z| + (1 + |z|^2)^{1/2}} |f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \, dt.$$

Reference

MACINTYRE, A. J., and W. W. ROGOSINSKI: Some elementary inequalities in function theory. Edinburgh Math. Notes **35**, 1–3 (1945).

3.8.40 Let f be an analytic function in the region $\operatorname{Re} z \geq 0$, except possibly for poles on the axis of imaginaries. Furthermore, let

$$f(\bar{z}) = \overline{f(z)} \quad \text{and} \quad \operatorname{Re} f(z) \geq 0 \quad \text{in} \quad \operatorname{Re} z \geq 0.$$

Then, for $\operatorname{Re} z \geq 0$,

$$|f'(z)| \leq \frac{f(z) + f(\bar{z})}{z + \bar{z}}.$$

Reference

REZA, F. M.: A bound for the derivative of positive real functions. SIAM Review **4**, 40–42 (1962).

3.8.41 Let $f(z)$ be an analytic function in the circle $|z| \leq R$. Let $M(r) = \max_{|z|=r\leq R} |f(z)|$, $A(r) = \max_{|z|=r\leq R} \operatorname{Re} f(z)$. For $0 \leq |z| = r < R$, we have

$$|f(z)| \leq |f(0)| + \frac{2r}{R-r}\left(A(R) - \operatorname{Re} f(0)\right),$$

$$\operatorname{Re} f(z) \leq \operatorname{Re} f(0) + \frac{2r}{R-r}\left(A(R) - \operatorname{Re} f(0)\right),$$

$$|\operatorname{Im} f(z) - \operatorname{Im} f(0)| \leq \frac{2Rr}{R^2 - r^2}\left(A(R) - \operatorname{Re} f(0)\right).$$

Reference

RAJAGOPAL, C. T.: Carathéodory's inequality and allied results II. Math. Student **15**, 5–7 (1947).

3.8.42 Let $a_0, a_1, \ldots$ be real numbers such that $0 < \sum_{k=0}^{+\infty} a_k^2 < +\infty$. If $0 \le \alpha < 1$, then

$$\sum_{\substack{i,j=0 \\ (i+j \text{ even})}}^{+\infty} \frac{a_i a_j}{i+j+1-\alpha} \le \frac{\pi}{2\cos\frac{\pi\alpha}{2}} \sum_{k=0}^{+\infty} a_k^2,$$

with equality if and only if $0 < \alpha < 1$, $a_0 = c$, $a_k = 0$ for odd k and

$$a_k = c \binom{\frac{1}{2}(k-1-\alpha)}{\frac{k}{2}}$$

for even k, where c is an arbitrary real constant different from 0.

Remark. This inequality is obviously closely related to HILBERT's inequality (see 3.9.36).

Reference

HUBER, A.: On an inequality of Fejér and Riesz. Ann. of Math. **(2) 63**, 572–587 (1956).

3.8.43 Let f be an analytic function in the closed disk $|z| \le 1$. Then for every θ $(0 \le \theta < 2\pi)$ and $\alpha > 0$,

$$\int_{-\pi}^{+\pi} |f(e^{it})|^\alpha \, dt \ge 2 \int_{-1}^{+1} |f(re^{i\theta})|^\alpha \, dr,$$

with equality if and only if $f \equiv 0$. Constant 2 is the best possible.

Remark. This inequality is due to L. FEJÉR and F. RIESZ [1]. For its generalizations see [2]–[5].

References

1. FEJÉR, L., and F. RIESZ: Über einige funktionentheoretische Ungleichungen. Math. Z. **11**, 305–314 (1921).
2. FRAZER, H.: On regular functions. J. London Math. Soc. **9**, 90–94 (1934).
3. BECKENBACH, E. F.: On a theorem of Fejér and Riesz. J. London Math. Soc. **13**, 82–86 (1938).
4. LOZINSKIĬ, S.: On subharmonic functions and their applications to the theory of surfaces (Russian). Bull. Acad. Sci. URSS Sér. Math. **8**, 175–194 (1944).
5. HUBER, A.: On an inequality of Fejér and Riesz. Ann. of Math. **(2) 63**, 572–587 (1956).

3.8.44 Let f be a continuous complex-valued function of a real variable x on $[a, b]$. If θ denotes any real number, then

$$\operatorname{Re}\left(e^{i\theta}\int_a^b f(x)\,dx\right) = \int_a^b \operatorname{Re}\left(e^{i\theta} f(x)\right) dx \leq \int_a^b |f(x)|\,dx,$$

since for any complex number z the inequality $\operatorname{Re} z \leq |z|$ holds. Now, $\int_a^b f(x)\,dx$ is itself a complex number, i.e.,

$$\int_a^b f(x)\,dx = re^{it} \qquad (t \text{ real}; r > 0).$$

If $\theta = -t$, then

$$\operatorname{Re}\left(e^{i\theta}\int_a^b f(x)\,dx\right) = r = \left|\int_a^b f(x)\,dx\right|,$$

which establishes the inequality

(1) $$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \qquad (a < b).$$

We have tacitly supposed above that $\int_a^b f(x)\,dx \neq 0$, but (1) also holds if $\int_a^b f(x)\,dx = 0$.

This well known inequality can be used to deduce some interesting inequalities which would be difficult to prove otherwise.

Example. Taking, for instance $f(x) = 1/(x + \lambda i)^{n+1}$, where i is the imaginary unit, n a positive integer and λ a real number, we get

(2) $$\frac{1}{n}\frac{|(a+\lambda i)^n - (b+\lambda i)^n|}{(a^2+\lambda^2)^{n/2}(b^2+\lambda^2)^{n/2}} \leq \int_a^b \frac{1}{(x^2+\lambda^2)^{(n+1)/2}}\,dx,$$

where $a < b$.

For $n = 1$ and $\lambda \neq 0$, (2) becomes

(3) $$\frac{b-a}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)}} \leq \frac{1}{\lambda}\left(\operatorname{arc\,tan}\frac{b}{\lambda} - \operatorname{arc\,tan}\frac{a}{\lambda}\right) \qquad (a < b).$$

Remark. Inequality (3) was proposed by D. S. Mitrinović as Problem 5283 in Amer. Math. Monthly **72**, 428 (1965). A solution, based on some geometric considerations, was given by M. F. Neuts and M. N. Tata in the same journal, **73**, 424 (1966). It was noted also that S. U. Rangarajan has proved an inequality in the other direction, namely

$$\operatorname{arc\,tan} b - \operatorname{arc\,tan} a < \frac{\pi}{2}\frac{b-a}{\sqrt{(a^2+1)(b^2+1)}},$$

where $a < b$ and $ab > -1$.

3.8.45 If a rectifiable curve L lies inside another convex closed curve C and f is a regular function inside C, then an absolute constant A exists so that

$$\int_L |f(z)|\,|dz| < A \int_C |f(z)|\,|dz|. \tag{1}$$

This is called GABRIEL's problem.

The best possible constant A' is not known, but it has been conjectured that $A' = 2$, which has been verified in the case when C is a circle.

There exits a very simple proof that (1) holds with $A = 4$. A refinement of the argument yields $A = 3.6$.

References

GABRIEL, R. M.: Some results concerning the integrals of moduli of regular functions along curves of certain types. Proc. London Math. Soc. **(2) 28**, 121–127 (1928).

GABRIEL, R. M.: Concerning integrals of moduli of regular functions along convex curves. Proc. London Math. Soc. **(2)** 39, 216–231 (1935).

CARLSON, F.: Quelques inégalités concernant les fonctions analytiques. Ark. Mat. Astr. Fys. **29 B**, No. **11** (1943), 6 pp.

ANDERSSON, B.: On an inequality concerning the integrals of moduli of regular analytic functions. Ark. Mat. **1**, No. 27, 367–373 (1951).

3.8.46 If $0 < p < q$ and $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a complex polynomial of degree n, then

$$\left(\frac{1}{2\pi}\int_{|z|=1} |P(z)|^q\,|dz|\right)^{1/q} < A_{p,q} n^{1/p - 1/q} \left(\frac{1}{2\pi}\int_{|z|=1} |P(z)|^p\,|dz|\right)^{1/p},$$

where $A_{p,q}$ is a constant which depends on p, q only.

References

NIKOL'SKIĬ, S. M.: Some inequalities for entire functions of finite degree of several variables and their applications (Russian). Doklady Akad. Nauk SSSR (N.S.) **76**, 785–788 (1951).

SZEGÖ, G., and A. ZYGMUND: On certain mean values of polynomials. J. Analyse Math. **3**, 225–244 (1953/54).

3.9 Miscellaneous Inequalities

3.9.1 If $a, b, c, d \geq 0$ and $c + d \leq \min(a, b)$, then

$$ad + bc \leq ab \quad \text{and} \quad ac + bd \leq ab.$$

3.9.2 Let c_k be the k-th elementary symmetric function of real variables $x_1, \ldots, x_n$. The two systems of inequalities

$$c_1 > 0, \ldots, c_n > 0 \tag{1}$$

and

(2) $$x_1 > 0, \ldots, x_n > 0$$

imply each other.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. In order to prove that (1) $\Rightarrow$ (2), we shall consider the algebraic equation

(3) $$x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^n c_n = 0,$$

whose roots are $x_1, \ldots, x_n$. After multiplication by $(-1)^n$, that equation leads to

$$(-x)^n + c_1(-x)^{n-1} + c_2(-x)^{n-2} + \cdots + c_n = 0.$$

Since $c_k > 0$, it follows that all the roots of this equation are positive (we use the known fact that all the roots are real). This proves the above statement.

Reference

Mitrinović, D. S.: Equivalence of two sets of inequalities. Math. Gaz. **43**, 126 (1959).

3.9.3 If a and b are positive numbers, then, for x, y real and $x^2 + y^2 > 0$,

$$\min(a, b) \le \frac{a\,|x| + b\,|y|}{\sqrt{x^2 + y^2}} \le \sqrt{a^2 + b^2}.$$

Reference

Ostrowski **2**, p. 290.

3.9.4 For $x^2 + y^2 + z^2 > 0$, we have

$$2 \le \frac{\sqrt{y^2 + z^2} + \sqrt{z^2 + x^2} + \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \le \sqrt{6},$$

where the lower bound is attained if and only if precisely one of x, y, z is non-zero, and the upper bound is attained if and only if $x = y = z$.

Reference

Ostrowski **2**, p. 290.

3.9.5 Let $b_1, \ldots, b_n$ be real numbers, let $a_1 \ge \cdots \ge a_n \ge 0$, and let

(1) $$\sum_{r=1}^{k} a_r \le \sum_{r=1}^{k} b_r \qquad (k = 1, \ldots, n).$$

Then

(2) $$\sum_{r=1}^{n} a_r^2 \le \sum_{r=1}^{n} b_r^2.$$

Proof. After multiplication by $a_k - a_{k+1}$, (1) becomes

$$(3) \qquad (a_k - a_{k+1}) \sum_{r=1}^{k} a_r \leq (a_k - a_{k+1}) \sum_{r=1}^{k} b_r \qquad (k = 1, \ldots, n),$$

where $a_{n+1} = 0$. Summing both sides of (3) from $k = 1$ to $k = n$, we obtain

$$(4) \qquad \sum_{r=1}^{n} a_r^2 \leq \sum_{r=1}^{n} a_r b_r.$$

By CAUCHY's inequality,

$$\left(\sum_{r=1}^{n} a_r b_r\right)^2 \leq \left(\sum_{r=1}^{n} a_r^2\right)\left(\sum_{r=1}^{n} b_r^2\right),$$

which with inequality (4) yields

$$\left(\sum_{r=1}^{n} a_r^2\right)^2 \leq \left(\sum_{r=1}^{n} a_r b_r\right)^2 \leq \left(\sum_{r=1}^{n} a_r^2\right)\left(\sum_{r=1}^{n} b_r^2\right),$$

so (2) follows.

Equality holds in (2) if and only if $a_r = b_r$ for $r = 1, \ldots, n$.

3.9.6 Assume that

1° $a_i, b_i, c_i, d_i \quad (i = 1, \ldots, n)$

are all positive numbers;

2° $\Sigma a_i \geq \Sigma c_i$;

3° $a_i - c_i = b_i - d_i \quad (i = 1, \ldots, n)$.

Then

$$\frac{\Sigma a_i \, \Sigma b_i}{\Sigma c_i \, \Sigma d_i} \leq \max\left(\frac{a_1 b_1}{c_1 d_1}, \ldots, \frac{a_n b_n}{c_n d_n}\right).$$

In order that

$$\frac{\Sigma a_i \, \Sigma b_i}{\Sigma c_i \, \Sigma d_i} \geq \min\left(\frac{a_1 b_1}{c_1 d_1}, \ldots, \frac{a_n b_n}{c_n d_n}\right),$$

condition 2° would have to be replaced by $\Sigma a_i = \Sigma c_i$.

Remark. These inequalities for a pseudo-mean were given by C. S. PATLAK and proved by D. C. B. MARSH. See Problem E 1449. Amer. Math. Monthly **68**, **670—671** (1961).

3.9.7 If a and b are real numbers and $r \geq 0$, then

$$(1) \qquad |a + b|^r \leq c_r (|a|^r + |b|^r),$$

where

$$c_r = 1 \quad \text{for} \quad r \leq 1 \quad \text{and} \quad c_r = 2^{r-1} \quad \text{for} \quad r > 1.$$

Proof. Since $x \mapsto f(x) = |x|^r$ $(r > 1)$ is a convex function, we get

$$\left|\frac{a+b}{2}\right|^r \leq \frac{1}{2}\left(|a|^r + |b|^r\right),$$

whence

$$|a+b|^r \leq 2^{r-1}\left(|a|^r + |b|^r\right).$$

For $r = 1$, inequality (1) becomes

$$|a+b| \leq |a| + |b|.$$

Now, let $0 \leq r < 1$. If a and b have opposite signs, the result is evidently true. Otherwise, let $t = b/a$ with $a \neq 0$. Then inequality (1) becomes

$$(1+t)^r \leq 1 + t^r \qquad (0 \leq r < 1).$$

For $r = 0$ this result is trivial. Consider the function

$$f(t) = (1+t)^r - 1 - t^r \qquad (0 < r < 1),$$

which vanishes at $t = 0$ and decreases as t increases. This yields inequality (1) for $a \neq 0$. (1) also holds when $a = 0$.

This proves inequality (1) which is important in the Theory of Probability.

Remark. Inequality (1) is true also for a and b complex numbers. See, for example, Mitrinović 2, p. 99, and R. Shantaram: Problem E 2147. Amer. Math. Monthly **76**, 1072–1073 (1969).

3.9.8 Let a_1, a_2, a_3 be positive numbers and let

(1) $\quad \min(a_1, a_2, a_3) < c_k < \max(a_1, a_2, a_3) \quad$ for $\quad k = 1, 2, 3.$

1° If (1) and $c_1 + c_2 + c_3 \geq a_1 + a_2 + a_3$ hold, then necessarily

$$c_1c_2c_3 \geq a_1a_2a_3,$$

$$c_2c_3 + c_3c_1 + c_1c_2 \geq a_2a_3 + a_3a_1 + a_1a_2.$$

Equality occurs if and only if the a_k's and c_k's in some order are equal.

2° If (1) and $a_1a_2a_3 \geq c_1c_2c_3$ hold, then necessarily

$$a_1 + a_2 + a_3 \geq c_1 + c_2 + c_3$$

(and indeed

$$a_1^m + a_2^m + a_3^m \geq c_1^m + c_2^m + c_3^m \qquad (m > 0)),$$

with equality if and only if the a_k's and c_k's in some order are equal.

3° If (1) and $a_2a_3 + a_3a_1 + a_1a_2 \geq c_2c_3 + c_3c_1 + c_1c_2$ hold, then necessarily

$$a_1 + a_2 + a_3 \geq c_1 + c_2 + c_3,$$

with equality if and only if the a_k's and c_k's in some order are equal.

The above result is due to A. Oppenheim [1]. He recently showed (see [2]) that the first inequality in 1° can be strengthened, which is not the case with 2°. Namely, he proved:

1'. Suppose that $0 \leq n \leq 2$, that the c_i's, a_i's satisfy (1) and that $c_1 + c_2 + c_3 \geq a_1 + a_2 + a_3$. Then

$$(a_1 + a_2 + a_3)^n \, c_1c_2c_3 \geq (c_1 + c_2 + c_3)^n \, a_1a_2a_3;$$

equality implies equality of the c_i's and a_i's.

For any $n > 2$ the inequality fails for appropriately chosen a_i's, c_i's.

2'. Let δ be an arbitrarily small positive number. Numbers a_i, c satisfying (1) and $a_1a_2a_3 > c_1c_2c_3$ exist such that

$$(a_1 + a_2 + a_3)\,(c_1c_2c_3)^\delta < (c_1 + c_2 + c_3)\,(a_1a_2a_3)^\delta$$

although (by 2°)

$$a_1 + a_2 + a_3 \geq c_1 + c_2 + c_3.$$

References.

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2. Oppenheim, A.: On inequalities connecting arithmetic means and geometric means of two sets of three positive numbers II. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **210–228**, **21–24** (1968).

3.9.9 If $a_1, \ldots, a_n$ $(n > 1)$ are different real numbers, then

$$\min_{1 \leq i < k \leq n} (a_k - a_i)^2 \leq \frac{12}{n(n-1)(n+1)} (a_1^2 + \cdots + a_n^2). \tag{1}$$

Proof. Let $a_1 < \cdots < a_n$, $\min_{1 \leq i < k \leq n} (a_k - a_i)^2 = d^2$ $(d > 0)$, $\min_i a_i^2 = a_j^2$, where j is a fixed index. Then, for $i > j$, $a_i > 0$, while for $i < j$, $a \ < 0$.

For $i = 1, \ldots, n$, put $b_i = a_j + (i - j)\,d$ which implies $b_i = b_1 + (i - 1)\,d$.

Then, for $i > j$,

$$\begin{aligned} a_i &= (a_i - a_{i-1}) + \cdots + (a_{j+1} - a_j) + a_j \geq (i - j)\,d + a_j \\ &= b_i \geq a_j \geq -a_i, \end{aligned} \tag{2}$$

and similarly for $i < j$,

$$a_i \leq b_i \leq -a_i. \tag{3}$$

From (2) and (3) we have

$$a_i^2 \geq b_i^2,$$

i.e.,

$$\sum_{i=1}^n a_i^2 \geq \sum_{i=1}^n b_i^2 = \sum_{i=1}^n \left(b_1^2 + 2b_1d(i-1) + (i-1)^2 d^2\right).$$

Whence, by the formula for the sum of squares of natural numbers, we have

$$\sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} b_i^2 = n\left(b_1 + \frac{n-1}{2}d\right)^2 + \frac{n(n-1)(n+1)}{12}d^2$$
$$\geq \frac{n(n-1)(n+1)}{12}d^2,$$

which implies inequality (1).

This proof is due to J. Polajnar. Another proof of (1), by indirect method, is given in Amer. Math. Monthly **76**, 691–692 (1969), as an answer to Problem E **2032** proposed by D. S. Mitrinović.

Comment by S. B. Prešić. From (1) follows that the inequality

$$\min_{1\leq i<k\leq n} (a_k - a_i)^2 \leq \frac{12}{n(n^2-1)} \sum_{\nu=1}^{n} (a_\nu + t)^2$$

holds for any real number t.

Since the function f defined by $f(t) = \sum_{\nu=1}^{n} (a_\nu + t)^2$ attains its minimum $\sum_{\nu=1}^{n} a_\nu - \frac{1}{n}\left(\sum_{\nu=1}^{n} a_\nu\right)^2$ for $t = -\frac{1}{n}\sum_{\nu=1}^{n} a_\nu$, we get the inequality

$$\min_{1\leq i<k\leq n} (a_k - a_i)^2 \leq \frac{12}{n(n^2-1)}\left(\sum_{\nu=1}^{n} a_\nu^2 - \frac{1}{n}\left(\sum_{\nu=1}^{n} a_\nu\right)^2\right),$$

which is stronger than (1).

3.9.10 Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two finite sets of real numbers with

$$A = \max_k (a_k + b_k) \quad \text{and} \quad B = \min_k (a_k + b_k).$$

Furthermore, let $i_1, \dots, i_n$ and $j_1, \dots, j_n$ be permutations of $1, \dots, n$ such that

$$a_{i_1} \geq \cdots \geq a_{i_n} \quad \text{and} \quad b_{j_1} \leq \cdots \leq b_{j_n}.$$

If $\overline{A} = \max_k (a_{i_k} + b_{j_k})$ and $\overline{B} = \min_k (a_{i_k} + b_{j_k})$, then

$$B \leq \overline{B} \leq \overline{A} \leq A. \tag{1}$$

Proof. Clearly, $\min_{i_k} (a_{i_k} + b_{i_k}) = B$. In the sequence of sums $\{a_{i_k} + b_{i_k}\}$ consider $a_{i_1} + b_{i_1}$ and $a_{j_1} + b_{j_1}$. If they are replaced by $a_{i_1} + b_{j_1}$ and $a_{j_1} + b_{i_1}$, the minimum of the new sequence is not less than B, since the new sums are not less than the smaller of the removed sums.

Next, in the same way, replace the summands in the sums which contain a_{i_2} and a_{j_2}, and so on. At each step the minimum of the sequence cannot decrease. After k such steps the sequence $\{a_{i_k} + b_{i_k}\}$ transforms into the sequence $\{a_{i_k} + b_{j_k}\}$, and thus we obtain $B \leq \overline{B}$.

Analogously we prove $\overline{A} \leq A$. Since $\overline{B} \leq \overline{A}$, inequality (1) is, therefore, proved.

This proof is due to G. KALAJDŽIĆ.

Remark. Inequality (1) was proposed as Problem 50 in Matematičeskoe Prosveščenie **6**, 330 (1961). However, this journal does not appear any more, this issue being the last published. So, no solution has been published in that journal.

3.9.11 Let a_{ij} and x_i ($i = 1, \ldots, k$; $j = 1, \ldots, n$) be nonnegative real numbers such that the numbers a_{ij} are not all equal, and let

$$\sum_{r=1}^{k} x_r = 1, \qquad \text{and} \qquad \min_i \left(\prod_{j=1}^{n} a_{ij}\right) = \prod_{j=1}^{n} a_{\mu j}, \qquad \max_i \left(\prod_{j=1}^{n} a_{ij}\right) = \prod_{j=1}^{n} a_{\nu j}$$

for $i = 1, \ldots, k$. Then

$$(1') \qquad \prod_{j=1}^{n} a_{\mu j} \leq \prod_{j=1}^{n} \left(\sum_{i=1}^{k} x_i a_{ij}\right),$$

with equality if and only if $x_\mu = 1$, $x_i = 0$ for $i \neq \mu$.

In addition, the inequality

$$(1'') \qquad \prod_{j=1}^{n} \left(\sum_{i=1}^{k} x_i a_{ij}\right) \leq \prod_{j=1}^{n} a_{\nu j}$$

holds if and only if

$$(2) \qquad \left(\sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{\nu j}}\right) \left(\sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{ij}}\right) \geq 0 \quad \text{for } i = 1, \ldots, k.$$

In this case, equality is attained in (1'') if and only if $x_\nu = 1$, $x_i = 0$ for $i \neq \nu$.

Proof. We have

$$(3) \qquad 0 \geq n - n \left(\prod_{j=1}^{n} \frac{a_{\nu j}}{a_{ij}}\right)^{1/n} \geq n - \sum_{j=1}^{n} \frac{a_{\nu j}}{a_{ij}} = \sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{ij}},$$

and, if we assume that (2) holds, then

$$(4) \qquad \sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{\nu j}} \leq 0 \qquad (i = 1, \ldots, k).$$

Further,

$$(5) \qquad \sum_{i=1}^{n} x_i a_{ij} \geq 0, \quad x_i \geq 0 \quad (j = 1, \ldots, n).$$

Putting $x_\nu = 1 - \left(\sum_{i=1}^{k} x_i - x_\nu\right)$, from (4) and (5) we find

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{k} x_i a_{ij}\right) = \prod_{j=1}^{n} a_{\nu j} \left(\prod_{j=1}^{n} \left(\sum_{i=1}^{k} x_i \frac{a_{ij} - a_{\nu j}}{a_{\nu j}}\right) + 1\right)$$
$$\leq \prod_{j=1}^{n} a_{\nu j} \left(\frac{1}{n} \sum_{i=1}^{k} x_i \left(\sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{\nu j}}\right) + 1\right)^n \leq \prod_{j=1}^{n} a_{\nu j}.$$

Conversely, suppose inequality (1'') holds, and consider the function

$$f(x_1, \ldots, x_k) = \prod_{j=1}^{n} a_{\nu j}\left(\sum_{i=1}^{k} x_i \frac{a_{ij} - a_{\nu j}}{a_{\nu j}} + 1\right).$$

For $x_i \in [0, 1]$ $(i = 1, \ldots, k)$, f assumes its greatest value at the point with coordinates $x_\nu = 1$, $x_i = 0$ for $i \neq \nu$ $(i = 1, \ldots, k)$. This means that the partial derivatives of the first order must be nonpositive; i.e.,

$$(6) \qquad \left(\frac{\partial f}{\partial x_i}\right)_{\substack{x_i=0\\ x_\nu=1}} = \sum_{j=1}^{n} \frac{a_{ij} - a_{\nu j}}{a_{\nu j}} \leq 0 \qquad (i = 1, \ldots, k;\ \ i \neq \nu).$$

From (3) and (6) we obtain (2).

Let us now prove inequality (1'). Putting $x_\mu = 1 - \left(\sum_{i=1}^{k} x_i - x_\mu\right)$, by the inequality between the weighted arithmetic and geometric means of positive numbers, we have

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{k} x_i a_{ij}\right) \geq \prod_{j=1}^{n}\prod_{i=1}^{k} a_{ij}^{x_i} = \prod_{j=1}^{n} a_{\mu j} \prod_{i=1}^{k}\left(\prod_{j=1}^{n}\frac{a_{ij}}{a_{\mu j}}\right)^{x_i} \geq \prod_{j=1}^{n} a_{\mu j}.$$

If all the numbers a_{ij} $(j = 1, \ldots, n)$ are equal, then (1') and (1'') reduce to equalities.

Remark. The above result is due to G. Kalajdžić and is a development and a generalization of an idea suggested by Problem E 2113 by F. Sand published in Amer. Math. Monthly **75**, 780 (1968).

3.9.12 Let a and $x_1, \ldots, x_n$ be positive numbers such that $\sum_{i=1}^{n} x_i = 1$. Then

$$\prod_{i_1<\cdots<i_k}\left(\sum_{j=1}^{k} x_{i_j}\right) \leq \left(\frac{k}{n}\right)^{\binom{n}{k}},$$

$$\prod_{i_1<\cdots<i_k}\left(\sum_{j=1}^{k}(1 - x_{i_j})\right) \leq \left(\frac{(n-1)k}{n}\right)^{\binom{n}{k}},$$

$$\sum_{i_1<\cdots<i_k}\left(\prod_{j=1}^{k}\frac{1 - x_{i_j}}{x_{i_j}}\right)^{a} \geq \binom{n}{k}(n-1)^{ka},$$

$$\sum_{i_1<\cdots<i_k}\left(\prod_{j=1}^{k}\frac{1}{1 - x_{i_j}}\right)^{a} \geq \binom{n}{k}\left(\frac{n}{n-1}\right)^{ka},$$

$$\sum_{i_1<\cdots<i_k}\left(\prod_{j=1}^{k}\frac{1}{x_{i_j}}\right)^{a} \geq \binom{n}{k} n^{ka}.$$

Equality holds in all these cases if and only if $x_1 = \cdots = x_n = 1/n$.

The above inequalities are due to V. Volenec, who, starting with these inequalities, gave a number of geometric inequalities which generalize some results of H. Gabai, J. Schopp, V. Thébault and Ž. Živanović.

3.9.13 If $a_k > 0$ for $k = 1, \ldots, n$, then

$$\sum_{k=1}^{n} (k!)^{1/k} \frac{(a_1 \cdots a_k)^{1/k}}{k+1} < \sum_{k=1}^{n} a_k .$$

Reference

Åkerberg, B.: A variant on the proofs of some inequalities. Proc. Cambridge Phil. Soc. **57**, 184–186 (1961).

3.9.14 Let $a_1, \ldots, a_n$ and $\alpha_1, \ldots, \alpha_n$ be real numbers such that

$$\begin{aligned} \alpha_1 &= a_1 + \cdots + a_n, \\ \alpha_1\alpha_2 &= a_2 + \cdots + a_n, \\ &\vdots \\ \alpha_1 \cdots \alpha_{n-1} &= a_{n-1} + a_n, \\ \alpha_1 \cdots \alpha_n &= a_n. \end{aligned} \tag{1}$$

Then the system of inequalities

$$0 < \alpha_k < 1 \qquad (k = 1, \ldots, n) \tag{2}$$

is equivalent to

$$a_1 + \cdots + a_n < 1, \quad a_k > 0 \quad (k = 1, \ldots, n). \tag{3}$$

Proof. From (1) we obtain

$$\begin{aligned} a_n &= \alpha_1 \cdots \alpha_n, \\ a_{n-1} &= \alpha_1 \cdots \alpha_{n-1}(1 - \alpha_n), \\ a_{n-2} &= \alpha_1 \cdots \alpha_{n-2}(1 - \alpha_{n-1}), \\ &\vdots \\ a_1 &= \alpha_1(1 - \alpha_2), \end{aligned} \tag{4}$$

and also

$$\begin{aligned} \alpha_1 &= a_1 + \cdots + a_n, \\ \alpha_2 &= \frac{a_2 + \cdots + a_n}{a_1 + \cdots + a_n}, \\ \alpha_3 &= \frac{a_3 + \cdots + a_n}{a_2 + \cdots + a_n}, \\ &\vdots \\ \alpha_n &= \frac{a_n}{a_{n-1} + a_n}. \end{aligned} \tag{5}$$

From (4) we infer that (2) implies that $a_k > 0$ $(k = 1, \ldots, n)$ and $a_1 + \cdots + a_n < 1$, since $\alpha_1 = a_1 + \cdots + a_n$.

From (5) we infer that (3) implies (2).

3.9.15 Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers and let

$$\alpha = n^{-1/2} \Sigma a_i \quad \text{and} \quad \beta = n^{-1/2} \Sigma b_i .$$

Then

$$\Sigma a_i^2 \, \Sigma b_i^2 - (\Sigma a_i b_i)^2 \geq \alpha^2 \, \Sigma b_i^2 - 2\alpha\beta \, \Sigma a_i b_i + \beta^2 \, \Sigma a_i^2 , \tag{1}$$

with equality if and only if the three row vectors

$$a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n) \quad \text{and} \quad e = n^{-1/2} \, (1, \ldots, 1)$$

are linearly dependent.

Proof. If A is a real $m \times n$ matrix, then the matrix $A'A$ of type $n \times n$ is semidefinite positive, i.e., $\det A'A \geq 0$, with equality if and only if $\operatorname{rank} A < \min(m, n)$.

Now, let

$$A = \begin{pmatrix} a_1 & b_1 & n^{-1/2} \\ \vdots & \vdots & \vdots \\ a_n & b_n & n^{-1/2} \end{pmatrix}.$$

Then

$$\det A'A = \Sigma a_i^2 \, \Sigma b_i^2 - (\Sigma a_i b_i)^2 - (\alpha^2 \, \Sigma b_i^2 - 2\alpha\beta \, \Sigma a_i b_i + \beta^2 \, \Sigma a_i^2).$$

The condition $\det A'A \geq 0$ reduces to (1), with equality if and only if $\operatorname{rank} A < \min(n, 3)$, i.e., if and only if the vectors a, b, e are linearly dependent.

References

SCHWERDTFEGER, H.: Problem 19. Canad. Math. Bull. 3, 79 (1960).

SCHWERDTFEGER, H.: Introduction to Linear Algebra and the Theory of Matrices. 2nd ed., Groningen 1961, p. 142.

Remark. Inequality (1) is an improvement of CAUCHY's inequality. For, the right side of (1) is ≥ 0, because the discriminant of the quadratic form in α, β is not positive, by CAUCHY's inequality.

3.9.16 Let $a_1, \ldots, a_n$ be nonnegative real numbers and

$$A_k = A - (n - 1) \, a_k \qquad (k = 1, \ldots, n) \tag{1}$$

with

$$A = a_1 + \cdots + a_n .$$

If all $A_1, \ldots, A_n$ are also nonnegative, then

$$a_1 \cdots a_n \geq A_1 \cdots A_n , \tag{2}$$

i.e.,

$$a_1 \cdots a_n \geq [(a_1 + \cdots + a_n) - (n-1)\, a_1] \\ \cdots [(a_1 + \cdots + a_n) - (n-1)\, a_n].$$

Proof. From (1) we find that, for $k = 1, \ldots, n$,

$$a_k = \frac{A_1 + \cdots + A_{k-1} + A_{k+1} + \cdots + A_n}{n-1}.$$

By the arithmetic-geometric mean inequality we have

(3) $$a_k \geq (A_1 \cdots A_{k-1} A_{k+1} \cdots A_n)^{1/(n-1)}.$$

Multiplying inequalities (3) for $k = 1, \ldots, n$, we obtain (2).

Equality will hold in (2) if and only if there is equality in (3) for all k or if some a_k is zero. Hence, we have equality in (2) if and only if $a_1 = \cdots = a_n \geq 0$, or if all $a_1, \ldots, a_n$ are equal except one which is zero.

If $n = 3$ we can omit the condition concerning A_k. If $n > 3$ the conditions $A_k \geq 0$ are essential. This can be seen by taking $a_1 = a_2 = 1$, $a_3 = a_4 = \cdots = a_n = \varepsilon > 0$, where ε is sufficiently small.

Remark. The above proof is revised from a proof by D. D. Adamović.

Reference

Mitrinović, D. S.: Problem 46. Mat. Vesnik 3 (18), 218–220 (1966).

3.9.17 If $x_1, \ldots, x_n$ are real numbers such that

$$\sum_{k=1}^{n} |x_k| = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_k = 0,$$

then

(1) $$\left|\sum_{k=1}^{n} a_k x_k\right| \leq \frac{1}{2}\left(\max_{1 \leq k \leq n} a_k - \min_{1 \leq k \leq n} a_k\right),$$

where $a_1, \ldots, a_n$ are real numbers.

Proof. We can assume, with no loss of generality, that

$$\min_{1 \leq k \leq n} a_k = a_1 \quad \text{and} \quad \max_{1 \leq k \leq n} a_k = a_n.$$

Then, in view of the hypothesis $\sum_{k=1}^{n} x_k = 0$, we have

(2) $$\left|\sum_{k=1}^{n} a_k x_k\right| = \frac{1}{2}\left|\sum_{k=1}^{n} (2a_k - a_n - a_1)\, x_k\right| \leq \frac{1}{2} \sum_{k=1}^{n} |2a_k - a_n - a_1|\, |x_k|.$$

Since the inequality $|2a_k - a_n - a_1| \leq a_n - a_1$ is equivalent to

$$a_k^2 - (a_1 + a_n)\, a_k + a_1 a_n \leq 0 \qquad (k = 1, \ldots, n),$$

which is true for $a_1 \leq a_k \leq a_n$ and, furthermore, since $\sum_{k=1}^{n} x_k = 1$, inequality (2) yields

$$\left|\sum_{k=1}^{n} a_k x_k\right| \leq \frac{1}{2} \sum_{k=1}^{n} (a_n - a_1) |x_k| = \frac{1}{2}\left(\max_{1\leq k\leq n} a_k - \min_{1\leq k\leq n} a_k\right).$$

This completes the proof of (1).

The above proof is due to G. Kalajdžić.

3.9.18 Let $\alpha_m = \frac{1}{n} \sum_{i=1}^{n} x_i^m$, subject to the conditions $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = n$, where $x_1, \ldots, x_n$ are real numbers.

α_3 and α_4 especially play an important part in skewness and kurtosis.

The following inequality

$$\alpha_4 \geq \alpha_3^2 + 1 \tag{1}$$

seems to have first been stated by K. Pearson [1]. Inequality (1) has later been proved by different methods by J. E. Wilkins [2] and M. C. Chakrabarti [3]. In [2] J. E. Wilkins also established that

$$\alpha_3 \leq \frac{n-2}{\sqrt{n-1}} \tag{2}$$

and that this value is attained at the point P:

$$x_1 = \sqrt{n-1}, \quad x_2 = \cdots = x_n = -\frac{1}{\sqrt{n-1}}.$$

M. C. Chakrabarti also proved inequality (2) using Sturm's theorem, and then showed that

$$\alpha_4 \leq n - 2 + \frac{1}{n-1}$$

and that this value is attained at the same point P.

All the above results are contained in an interesting paper [4] of M. Lakshmanamurti. He proved that

$$\alpha_{2m} \geq \alpha_{m+1}^2 + \alpha_m^2$$

giving the conditions of equality, and also showed that

$$\alpha_m \leq \frac{(n-1)^{m-1} + (-1)^m}{n(n-1)^{(m/2)-1}},$$

where α_m attains its upper bound at the point P, where m need not be a positive integer.

References

1. Pearson, K.: Mathematical contributions to the theory of evolution. XIX: Second supplement to a memoir on skew variation. Phil. Trans. Roy. Soc. **A 216**, 432 (1916).
2. Wilkins, J. E.: A note on skewness and kurtosis. Ann. Math. Statistics **15**, 333–335 (1944).
3. Chakrabarti, M. C.: A note on skewness and kurtosis. Bull. Calcutta Math. Soc. **38**, 133–136 (1946).
4. Lakshmanamurti, M.: On the upper bound of $\sum_{i=1}^{n} x_i^m$ subject to the conditions $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = n$. Math. Student **18**, 111–116 (1950).

3.9.19 Consider a real sequence (a_n) such that

$$\Delta^\nu a_n > 0 \quad \text{for} \quad \nu = 0, 1, \ldots, k, \quad \Delta^0 a_1 = a_1, \quad \text{and} \quad \Delta^{k+1} a_n = 0,$$

where $\Delta^\nu a_k$ is as defined in 3.9.20.

If $S_n = a_1 + \cdots + a_n$, then

$$\frac{m+n}{n} < \frac{S_{m+n}}{S_n} < \frac{\binom{m+n}{k+1}}{\binom{n}{k+1}}.$$

Reference

Marković, D.: Quelques remarques sur les progressions arithmétiques (Serbian). Bull. Soc. Math. Phys. Serbie **1**, No. 2, 17–21 (1949).

3.9.20 Let $a_1, a_2, \ldots$ be a sequence of real numbers, and let

$$\Delta^0 a_k = a_k, \quad \Delta^1 a_k = \Delta a_k = a_{k+1} - a_k,$$

$$\Delta^\nu a_k = \Delta(\Delta^{\nu-1} a_k) \quad \text{for} \quad \nu = 2, 3, \ldots \quad \text{and} \quad k = 0, 1, \ldots.$$

If $b \geq -1$, $(-1)^n \Delta^n a_0 > 0$ and

$$(-1)^{n-m} \Delta^{n-m} a_m \geq 0 \quad \text{for} \quad m = 1, \ldots, n,$$

then

$$\sum_{k=0}^{n} \binom{n}{k} b^k a_k > 0.$$

If $b \leq -1$, $\Delta^n a_0 > 0$ and

$$\Delta^{n-m} a_m \geq 0 \quad \text{for} \quad m = 1, \ldots, n,$$

then

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} b^k a_k > 0.$$

Let $x \mapsto f_1(x), \ldots, f_q(x)$ be given continuous functions over the interval $[0, n]$, each differentiable n times on $(0, n)$ and $f_1(n) > 0, \ldots, f_q(n) > 0$. If, furthermore,

$$(-1)^k f_p^{(k)}(x) \geq 0$$

for $p = 1, \ldots, q$, $k = 1, \ldots, n$ and $n - k < x < n$, then

$$\sum_{r=0}^{n} \binom{n}{r} b^r \prod_{p=1}^{q} f_p(r) > 0,$$

where $b > -1$.

The above inequalities are due to M. P. Drazin.

Reference

Drazin, M. P.: Some inequalities arising from a generalized mean value theorem. Amer. Math. Monthly **62**, 226–232 (1955).

3.9.21 Generalizing some results of H. C. Pocklington [1], T. Nomura [2] has proved the following results:

1° Let $x_1, \ldots, x_n$ and a, b, k be positive numbers, and suppose $0 \leq y_1 \leq \cdots \leq y_n$. If

$$\sum_{r=1}^{n} x_r \leq ka, \quad \sum_{r=1}^{s} x_r \leq a \quad \text{and} \quad \sum_{r=1}^{t} y_r \leq b \quad \text{for} \quad n \geq s \geq t \geq k,$$

then

$$\sum_{r=1}^{n} x_r y_r \leq ab.$$

2° Let A and B be positive numbers. If $af(x_r) \leq Ax_r$, and $bg(y_r) \leq By_r$ $(r = 1, \ldots, n)$ in addition to the conditions of 1°, then

$$\sum_{r=1}^{n} f(x_r)\, g(y_r) \leq AB.$$

Proof of 1°. If we put $\sum_{r=1}^{m} x_r = X_m$, then

$$\begin{aligned}\sum_{r=1}^{n} x_r y_r &\leq y_1 X_s + y_s(X_n - X_s) = (y_1 - y_s)\, X_s + y_s X_n \\ &\leq (y_1 - y_s)\, a + y_s ka = [y_1 + y_s(k-1)]\, a \\ &\leq [y_1 + y_s(t-1)]\, a \\ &\leq (y_1 + y_2 + \cdots + y_t)\, a \\ &\leq ab.\end{aligned}$$

Proof of 2°. We have

$$\sum_{r=1}^{n} f(x_r)\, g(y_r) \leq \sum_{r=1}^{n} \frac{AB}{ab}\, x_r y_r = \frac{AB}{ab} \sum_{r=1}^{n} x_r y_r .$$

Using 1° we get

$$\sum_{r=1}^{n} f(x_r)\, g(y_r) \leq AB .$$

In his paper T. Nomura has also proved an analogous result in which the sequence $y_1, \ldots, y_n$ is substituted by several such sequences.

References

1. Pocklington, H. C.: An inequality. J. London Math. Soc. **10**, 242–243 (1935).
2. Nomura, T.: On some inequalities. Sci. Rep. Tokyo Bunrika Daigaku **A 3**, 181–183 (1937).

3.9.22 Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers such that

$a_1 \geq \cdots \geq a_n > 0$ and $b_1 \geq a_1,\ b_1 b_2 \geq a_1 a_2,\ \ldots,\ b_1 \cdots b_n \geq a_1 \cdots a_n$.

Then

$$b_1 + \cdots + b_n \geq a_1 + \cdots + a_n . \tag{1}$$

Under the same hypothesis we have the following inequality

$$\sum_{r=1}^{n} \frac{b_r - a_r}{a_n} \geq \sum_{r=1}^{n} \frac{b_r - a_r}{a_r} ,$$

which is stronger than (1).

Another possible generalization is the following. Put

$$\log a_r = \alpha_r, \quad \log b_r = \beta_r \quad \text{for} \quad r = 1, \ldots, n .$$

The condition $b_1 \geq a_1, \ldots, b_1 \cdots b_n \geq a_1 \cdots a_n$ and (1) take the form

$$\sum_{r=1}^{k} \beta_r \geq \sum_{r=1}^{k} \alpha_r \quad \text{and} \quad \sum_{r=1}^{k} e^{\beta_r} \geq \sum_{r=1}^{k} e^{\alpha_r}$$

for $k = 1, \ldots, n$.

More generally, if $\sum_{r=1}^{k} \beta_r \geq \sum_{r=1}^{k} \alpha_r$ for $k = 1, \ldots, n$ is true and f is a monotonely increasing convex function, then

$$\sum_{r=1}^{k} f(\beta_k) \geq \sum_{r=1}^{k} f(\alpha_k) \quad \text{for} \quad k = 1, \ldots, n .$$

Reference

Szász, G., L. Gehér, I. Kovács and L. Pintér: Contests in Higher Mathematics. Budapest 1968, p. 19 and pp. 145–148.

3.9.23 Let a and b be two arbitrary positive numbers satisfying the condition $1 < a < b$. If

$$u_1 = b, \quad v_1 = a, \quad u_{n+1} = \frac{1}{2}(u_n + v_n), \quad v_{n+1} = \frac{2u_n v_n}{u_n + v_n},$$

then

$$0 < u_{n+1} - v_{n+1} < \frac{(b-a)^{2^n}}{2^n}.$$

3.9.24 Let $a = (a_k)$ and $b = (b_k)$ be two sequences of real numbers such that:

1° $\sum_{k=1}^{m} a_k > 0$, for some m,

2° $a_j < 0$ $(j > 1) \Rightarrow a_{j-1} < 0$,

3° $1 \leq b_1 \leq \cdots \leq b_m$.

Then

$$\sum_{k=1}^{m} a_k b_k \geq \sum_{k=1}^{m} a_k. \tag{1}$$

Proof. If there are no negative terms in the sequence a up to the m-th term, there is nothing to prove. If there are negative terms, then there is some k_0 such that

$$a_{k_0} < 0, \quad a_k \geq 0 \quad \text{for} \quad k > k_0.$$

We now have

$$\sum_{k=1}^{m} a_k b_k \geq \sum_{k=1}^{m} a_k b_{k_0} = b_{k_0} \sum_{k=1}^{m} a_k \geq \sum_{k=1}^{m} a_k.$$

Obviously one single strict inequality in 3° implies strict inequality in (1).

Reference

LEWIN, M.: An inequality and its application to a property of Mahler's partition functions. J. London Math. Soc. **43**, 429–432 (1968).

3.9.25 Let $p_0 = 0, p_1, p_2, \ldots, p_{2n+1}$ be a sequence of real nonnegative numbers such that

$$p_{2k} \leq p_{2k-1} \quad \text{and} \quad p_{2k} \leq p_{2k+1} \quad \text{for} \quad k = 1, \ldots, n.$$

Then

$$\left(\sum_{k=0}^{n} (p_{2k+1} - p_{2k})\right)^2 + \sum_{k=1}^{n} p_{2k}^2 \geq \sum_{k=0}^{n} p_{2k+1}^2.$$

Reference

OPIAL, Z.: Sur une inégalité. Ann. Polon. Math. **8**, 29–32 (1960).

3.9.26 Let a_i, b_i $(i = 0, 1, \ldots, 2n)$ be real numbers such that

$$a_i + a_{i+1} \geq 0 \qquad (i = 0, 1, \ldots, 2n-1),$$
$$a_{2i+1} \leq 0 \qquad (i = 0, 1, \ldots, n-1),$$
$$\sum_{k=2p}^{2q} b_k > 0 \qquad (0 \leq p \leq q \leq n).$$

Then

$$\sum_{i=0}^{2n} (-1)^i a_i b_i \geq 0,$$

equality occurring if and only if $a_i = 0$ for all i.

Reference

MASSERA, J., and D. S. GREENSTEIN: Problem 4608. Amer. Math. Monthly **63**, 49 (1956).

3.9.27 Let $0 \leq x_0 < x_1 < \cdots < x_n \leq 1$ $(n \geq 1)$ and let

$$x_k - x_{k-1} > d > 0 \qquad (k = 1, \ldots, n).$$

Let j $(0 \leq j \leq n)$ be an integer and let $x \in [0, 1]$. Then

$$\prod_{\substack{k=0\\k \neq j}}^{n} \left| \frac{x - x_k}{x_j - x_k} \right| < \frac{\prod_{k=0}^{n-1} (1 - kd)}{j!\,(n-j)!\,d^n}.$$

Reference

SHISHA, O.: The Chebyshev polynomial of best approximation to a given function on an interval. Math. Comp. **20**, 267–268 (1966).

3.9.28 Let x, y, p, q be real numbers and $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|x|^{q/p} \operatorname{sgn} x - |y|^{q/p} \operatorname{sgn} y|^p \leq \max\left(2^p, \left(\frac{q}{p}\right)^p\right) |x - y|^p \left(|x|^{q-p} + |y|^{q-p}\right),$$

$$\left||x|^{p/q} \operatorname{sgn} x - |y|^{p/q} \operatorname{sgn} y\right|^q \leq 2^q\, |x - y| \left(|x|^{p-q} + |y|^{p-q}\right).$$

Reference

CITLANADZE, È. S.: On the variational theory of a class of non-linear operators in the space L_p $(p > 1)$ (Russian). Doklady Akad. Nauk SSSR (N.S.) **71**, 441–444 (1950).

3.9.29 Let a be a positive constant. Define f by $f(0) = f(1) = 0$, and

$$f(x) = \exp\left(-x^{-a} - (1-x)^{-a}\right) \quad \text{for} \quad 0 < x < 1.$$

Then

$$|f^{(n)}(x)| \leq (Cn^{1+a^{-1}})^n \quad \text{for} \quad n = 0, 1, \ldots, \quad \text{and} \quad 0 \leq x \leq 1,$$

where C depends on a only.

Reference

DE BRUIJN, N. G.: Problem 168. Wisk. Opgaven **19**, 357–358 (1950/54).

3.9.30 Given

$$(1 + z + \cdots + z^k)^n = c_0^{(k,n)} + c_1^{(k,n)} z + \cdots + c_{kn}^{(k,n)} z^{kn},$$

then, for kn odd,

$$c_0^{(k,n)} \leq c_1^{(k,n)} \leq \cdots \leq c_{(kn-1)/2}^{(k,n)} = c_{(kn+1)/2}^{(k,n)} \geq c_{(kn+3)/2}^{(k,n)} \geq \cdots \geq c_{kn}^{(k,n)}$$

and, for kn even,

$$c_0^{(k,n)} \leq c_1^{(k,n)} \leq \cdots \leq c_{kn/2}^{(k,n)} \geq c_{(kn/2)+1}^{(k,n)} \geq \cdots \geq c_{kn}^{(k,n)}.$$

Remark. We have

$$c_p^{(k,n+1)} = \sum_{i=\max(0,p-k)}^{p} c_i^{(k,n)},$$

$$c_p^{(k,2)} = p + 1 \quad \text{for} \quad 0 \leq p \leq k.$$

For $n \geq 2$,

$$c_p^{(k,n)} > c_{p-1}^{(k,n)} \quad \left(1 \leq p \leq \left[\frac{kn}{2}\right]; \quad c_{kn-p}^{(k,n)} = c_p^{(k,n)}\right).$$

References

TURÁN, P.: Problem 3950. Amer. Math. Monthly **47**, 182 (1940).
FRAME, J. S.: Solution of Problem 3950. Amer. Math. Monthly **48**, 562–563 (1941).

3.9.31 Let m and n, with $n < m$, be natural numbers, and let

$$f(x) = \frac{1 + x + \cdots + x^m}{1 + x + \cdots + x^n}.$$

Then:

1° f is strictly increasing on $[0, +\infty)$,

2° g defined by $g(x) = f(x) \cdot x^{n-m}$ is strictly decreasing on $(0, +\infty)$,

3° $1 < f(x) < \dfrac{m+1}{n+1} \qquad (0 < x < 1)$,

$$f(x) > \frac{m+1}{n+1} x^{m-n} \qquad (0 < x < 1),$$

$$f(x) > \frac{m+1}{n+1} \qquad (1 < x < +\infty),$$

$$1 < f(x) < \frac{m+1}{n+1} x^{m-n} \qquad (1 < x < +\infty).$$

Proof. Since

$$f(x) = 1 + \frac{x + x^2 + \cdots + x^{m-n}}{1 + \frac{1}{x} + \cdots + \frac{1}{x^n}},$$

result 1° immediately follows.

2° follows from 1° by noting that $g(1/x) = f(x)$.

From 1° and 2° we get

$$\begin{aligned} f(0) < f(x) < f(1) \qquad & (0 < x < 1), \\ g(x) > g(1) \qquad & (0 < x < 1), \\ f(x) > f(1) \qquad & (1 < x < +\infty), \\ g(+\infty) < g(x) < g(1) \qquad & (1 < x < +\infty). \end{aligned}$$

It turns out that these inequalities are equivalent to those in 3°.

Remark. Revised from a proof due to D. D. Adamović.

3.9.32 Let $n \geq 1$ be an integer and $0 \leq t \leq \pi$. Let $F_n(t)$ be defined by

$$F_n(t) = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} \sin x \left(\frac{t^{n+1}}{(n+1)!}\right)^{-1} dx,$$

$$F_n(0) = 1.$$

Then

(1) $$F_n'(t) < 0 \quad \text{for} \quad 0 < t \leq \pi \quad \text{and} \quad F_n'(0) = 0.$$

An application of (1) gives

$$F_n(\pi) \leq F_n(t) \leq F_n(0) = 1 \quad \text{for} \quad 0 \leq t \leq \pi \quad \text{and} \quad n \geq 1,$$

where

$$F_n(t) = (-1)^s \left[\cos t - \sum_{k=0}^{s-1} (-1)^k \frac{t^{2k}}{(2k)!}\right] \left(\frac{t^{n+1}}{(n+1)!}\right)^{-1},$$

if $n = 2s - 1$ and $s \geq 1$, and

$$F_n(t) = (-1)^s \left[\sin t - \sum_{k=0}^{s-1} (-1)^k \frac{t^{2k+1}}{(2k+1)!}\right] \left(\frac{t^{n+1}}{(n+1)!}\right)^{-1},$$

if $n = 2s$ and $s \geq 1$.

Some particular cases of the above result are the inequalities

$$\frac{\sin t}{t} \leq \frac{2(1 - \cos t)}{t^2}, \quad \frac{\sin t}{t} \leq \frac{2 + \cos t}{3}, \qquad v. \S 1, 5, 2$$

$$\frac{\sin t}{t} \geq \frac{4 - 4\cos t - t^2}{t^2}, \quad \frac{\sin t}{t} \geq \frac{\cos t + 4 - t^3/3}{5},$$

all valid for $0 \leq t \leq \pi$ and with equality only when $t = 0$.

Reference

Everitt, W. N.: A trigonometric inequality. Math. Gaz. **44**, 52–54 (1960).

3.9.33 If $\alpha_k \in E^n$, $|\alpha_k| \leq 1$ $(k = 1, \ldots, m)$ and

$$\left|\sum_{k=1}^{m} \alpha_k\right| = a,$$

then there exists a permutation σ of $1, \ldots, n$ such that

$$\left|\sum_{k=1}^{p} \alpha_{\sigma(k)}\right| \leq (a + 1)\, 5^{\frac{n-1}{2}} \text{ for all } p = 1, \ldots, m.$$

The above result can be found in [1].

Reference

1. Bourbaki, N.: Topologie générale. 2nd. ed., Paris 1955, Chap. 5–8, p. 87.

3.9.34 Let P be the set of all mappings of a set S into the nonnegative reals. Let M be a mapping of P into the nonnegative real numbers, satisfying

1° $M(0) = 0$, $M(\lambda f) = \lambda M(f)$,

where $\lambda > 0$ and $f \in P$;

2° $f(x) \leq g(x)$ for all $x \in S$ and $f, g \in P$ implies $M(f) \leq M(g)$;

3° $M(f + g) > M(f) + M(g)$ for all $f, g \in P$.

Let $h(t_1, \ldots, t_n)$ be a real-valued function of n real variables $t_1, \ldots, t_n$ defined and continuous for $t_i \geq 0$ $(i = 1, \ldots, n)$. Let h have the following properties:

4° Inequalities $t_i > 0$ $(i = 1, \ldots, n)$ imply that $h(t_1, \ldots, t_n) > 0$.

5° If $\lambda > 0$, then $h(\lambda t_1, \ldots, \lambda t_n) = \lambda h(t_1, \ldots, t_n)$.

6° The set $K \subset E^n$ of all points $(t_1, \ldots, t_n)$, whose coordinates satisfy

$$t_i \geq 0 \qquad (i = 1, \ldots, n)$$

and

$$h(t_1, \ldots, t_n) \geq 1,$$

is convex.

Under these conditions, if $f_1, \ldots, f_n \in P$, we have

$$M\big(h(f_1, \ldots, f_n)\big) \leq h\big(M(f_1), \ldots, M(f_n)\big). \tag{1}$$

If $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$ and $f, g \in P$, then

$$M(f^\alpha g^\beta) \leq M(f)^\alpha\, M(g)^\beta. \tag{2}$$

If $p \geq 1$ and $f, g \in P$, then

$$M\big((f + g)^p\big)^{1/p} \leq M(f^p)^{1/p} + M(g^p)^{1/p}. \tag{3}$$

Hölder's inequality (2) and Minkowski's inequality (3) are immediate consequences of (1). We have only to set $h(x, y) = x^\alpha y^\beta$ and $h(x, y) = (x^{1/p} + y^{1/p})^p$ respectively, and prove that K is a convex set in both cases.

Example. Let us take $S = \{1, \ldots, n\}$. Then P is the set of all nonnegative sequences $(a_1, \ldots, a_n)$ of real numbers. We can take

$$M(a_1, \ldots, a_n) = \frac{\sum_{k=1}^{n} w_k a_k}{\sum_{k=1}^{n} w_k},$$

where $w_k > 0$ for all k.

Inequalities (2) and (3) give

$$\sum_{k=1}^{n} w_k a_k^\alpha b_k^\beta \leq \left(\sum_{k=1}^{n} w_k a_k\right)^\alpha \left(\sum_{k=1}^{n} w_k b_k\right)^\beta \tag{4}$$

and

$$M_n^{[p]}(a + b; w) \leq M_n^{[p]}(a; w) + M_n^{[p]}(b; w), \tag{5}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$ and $p \geq 1$; besides, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ are nonnegative sequences and $w = (w_1, \ldots, w_n)$ is a positive sequence.

Reference

Bourbaki, N.: Intégration. Paris 1952, Chap. 1, No. 1, 2, pp. 9–14.

3.9.35 Let $m = \min(a, ax^{a-1})$ and $M = \max(a, ax^{a-1})$ for $x > 0$ and for positive rational a. Let H, G, A be the harmonic, geometric and arithmetic means of m and M, respectively. Then the quotient

$$\frac{x^a - 1}{x - 1} \quad \text{for} \quad x \neq 1$$

is contained in the intervals

$$(m, H),\ (H, G),\ (G, A),\ \text{or}\ (A, M)$$

in the following respective cases

$$\frac{1}{2} < a < 1,\quad 0 < a < \frac{1}{2},\quad a > 2 \text{ or } 1 < a < 2.$$

Reference

Iyengar, K. S. K.: A deepening of the binomial inequality. J. Mysore Univ. **B** 3, Part 19, 135–138 (1942).

Remark. G. Kalajdžić proved that the above result is valid also when a is an arbitrary positive real number.

3.9.36 Let $a_0, a_1, \ldots, a_N$ be positive real numbers. Then

$$(1)\qquad \sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_m a_n}{m+n+1} \le \pi \sum_{n=0}^{N} a_n^2.$$

Generalization. There are a number of extensions of (1) which is known as HILBERT's inequality [1].

For instance, H. FRAZER [2] demonstrated that the constant π can be replaced by a smaller constant $(N+1)\sin\frac{\pi}{N+1}$. E. H. COSPEY, H. FRAZER and W. W. SAWYER [3] proved that in certain cases even this constant can be lessened.

N. G. DE BRUIJN and H. S. WILF [4] showed that the best possible constant C_N in

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m a_n}{m+n} \le C_N \sum_{n=1}^{N} a_n^2$$

is given by

$$C_N = \pi - \frac{1}{2}\pi^5 (\log N)^{-2} + O\big(\log\log N (\log N)^{-3}\big) \quad (N \to +\infty).$$

F. C. HSIANG [5] proved that, if $a_0, a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b_N$ are sequences of nonnegative real numbers,

$$\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_n b_m}{2n+2m+1} \le (N+1)\sin\frac{\pi}{2(N+1)}\left(\sum_{n=0}^{N} a_n^2\right)^{1/2}\left(\sum_{n=0}^{N} b_n^2\right)^{1/2}.$$

Related to this inequality, see paper [6] of Q. A. M. M. YAHYA.

D. V. WIDDER [7] demonstrated the following inequality

$$\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_m a_n}{m+n+1} \le \pi \sum_{n=0}^{N}\sum_{m=0}^{N}\frac{(m+n)!}{m!\,n!}\,\frac{a_m a_n}{2^{m+n+1}},$$

which is stronger than that of HILBERT. HILBERT's inequality is also valid when $N \to +\infty$, under the assumption that $\sum_{n=0}^{+\infty} a_n^2$ converges. For this case as well as for its integral analogues, consult [8] and [9].

References

1. HILBERT, D.: Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. **32**, 342–350 (1888).
2. FRAZER, H.: Note on Hilbert's inequality. J. London Math. Soc. **21**, 7–9 (1946).
3. COSPEY, E. H., H. FRAZER and W. W. SAWYER: Empirical data on Hilbert's inequality. Nature (London) **161**, 361 (1948).
4. DE BRUIJN, N. G., and H. S. WILF: On Hilbert's inequality in n dimensions. Bull. Amer. Math. Soc. **68**, 70–73 (1962).
5. HSIANG, F. C.: An inequality for finite sequences. Math. Scand. **5**, 12–14 (1957).
6. YAHYA, Q. A. M. M.: On the generalization of Hilbert's inequality. Amer. Math. Monthly **72**, 518–520 (1965).

7. WIDDER, D. V.: An inequality related to one of Hilbert's. J. London Math. Soc. 4, 194–198 (1929).
8. GODUNOVA, E. K.: Generalization of two-parameter Hilbert's inequality (Russian). Izv. Vysš. Učebn. Zav. Mat. 1967, No. 1 (56), 35–39.
9. LEVIN, V. I., and S. B. STEČKIN: Inequalities. Amer. Math. Soc. Transl. (2) 14, 1–29 (1960).

3.9.37 Let f be a function of t and x defined for $\alpha \le t \le \beta$ and $a \le x \le b$, and continuous in this rectangle. For each fixed t, let f be positive, twice continuously differentiable, and logarithmically convex as a function of x. Then the function F defined by

$$F(x) = \int_\alpha^\beta f(t, x)\, dt$$

is logarithmically convex.

Reference

HILLE, E.: Analysis, vol. 2. Waltham-Toronto-London 1966, p. 521.

3.9.38 Let x_0 be an approximate value of $\sqrt{N}$, where N is a natural number. Then

$$x_1 = x_0 \frac{3N + x_0^2}{3x_0^2 + N}$$

is a better approximation of $\sqrt{N}$, in the sense that if $x_0 < \sqrt{N}$, then $x_0 < x_1 < \sqrt{N}$, while if $x_0 > \sqrt{N}$, then $x_0 > x_1 > \sqrt{N}$.

Similarly, if x_0 is an approximation of $\sqrt[3]{N}$, then

$$x_1 = x_0 \frac{2N + x_0^3}{2x_0^3 + N}$$

is a better approximation of $\sqrt[3]{N}$. If $x_0 < \sqrt[3]{N}$, then $x_0 < x_1 < \sqrt[3]{N}$, and if $x_0 > \sqrt[3]{N}$, then $\sqrt[3]{N} < x_1 < x_0$.

These approximations can be obtained from a generalization of NEWTON's method due to R. E. SHAFER, who privately communicated to us this result.

3.9.39 If $a_1, \ldots, a_n$ denote real numbers and if

$$\frac{1}{(1 - a_1 x) \cdots (1 - a_n x)} = \binom{n-1}{0} + \binom{n}{1} q_1 x + \cdots + \binom{n + r - 1}{r} q_r x^r + \cdots,$$

then, for $r = 0, 1, \ldots,$

$$q_{r+1}^2 < q_{2r} q_{2r+2} \quad \text{with} \quad q_0 = 1.$$

Reference

GHIRCOIAŞIU, N.: On the expansion into power series of the inverse of a polynomial (Romanian). Stud. Cerc. Mat. 6, 51–77 (1955).

3.9.40 If

$$S(a, n) = \sum_{k=n+1}^{+\infty} \frac{1}{k^2 + ak + 1},$$

then we have

$$S(a, n) < \frac{\pi}{\sqrt{4 - a^2}} - \frac{2}{\sqrt{4 - a^2}} \arctan \frac{2n + a}{\sqrt{4 - a^2}} \qquad (0 < a < 2),$$

$$S(a, n) < \frac{1}{n + 1} \qquad (a = 2),$$

$$S(a, n) < \frac{1}{\sqrt{a^2 - 4}} \log \frac{2n + a + \sqrt{a^2 - 4}}{2n + a - \sqrt{a^2 - 4}} \qquad (a > 2).$$

Proof. The quadratic polynomial $x^2 + ax + 1$ $(a > 0)$ is positive increasing for $x \geq 0$. It follows that $x \mapsto f(x) = 1/(x^2 + ax + 1)$ is a positive decreasing function for $x \geq 0$ and $f(x) \to 0$ as $x \to +\infty$. Therefore we have

$$\sum_{k=n+1}^{+\infty} \frac{1}{k^2 + ak + 1} < \int_n^{+\infty} f(x)\, dx. \tag{1}$$

When we look for a primitive function of f, we must distinguish three cases: $0 < a < 2$, $a = 2$, $a > 2$. Evaluating the definite integral on the right-hand side of (1), we obtain the corresponding upper bounds.

3.9.41 Let $F_n(x) = \frac{1}{n!}(1 - x) \cdots (n - x)$ for $n \geq 2$, and let $I(n) = \int_0^1 F_n(x)\, dx$. Then

$$I(n) > \frac{1}{2 \log 2n} \quad \text{for} \quad n \geq 2,$$

whence it follows that $\sum_{n=1}^{+\infty} \frac{1}{n} I(n)$ diverges.

This result is due to YA. GABOVIČ.

3.9.42 Let, for $x \geq 1$, the function f be decreasing and positive and assume that $\sum_{n=1}^{+\infty} f(n)$ converges. Let $a_1, \ldots, a_k$ be positive, and $a_1^{-1} + \cdots + a_k^{-1} = 1$. Then

$$\sum_{n=1}^{+\infty} f(na_1) + \cdots + \sum_{n=1}^{+\infty} f(na_k) \leq \sum_{n=1}^{+\infty} f(n).$$

Reference

DE BRUIJN, N. G.: Problem 51. Wisk. Opgaven 19, 104–105 (1950/54).

3.9.43 Assume $0 < a < 1$, $a^{-1} + b^{-1} = 1$, and let $[x]$ denote the integral part of x. Then

$$\sum_{n=1}^{+\infty}[na]^{-2} + \sum_{n=1}^{+\infty}[nb]^{-2} \leq \sum_{n=1}^{+\infty} n^{-2}.$$

Equality holds if and only if a is irrational.

Reference

DE BRUIJN, N. G.: Problem 52. Wisk. Opgaven **19**, 105–106 (1950/54).

Remark. Compare with 3.9.42.

3.9.44 If $p \geq 1$ and $\sum_{n=0}^{+\infty} |a_n|^p < +\infty$, then

$$\int_0^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{a_n e^{-x} x^n}{n!} \right|^p dx \leq \sum_{n=0}^{+\infty} |a_n|^p.$$

If $p > 1$ equality holds only if $a_0 = a_1 = \cdots = 0$. For no value of $p > 1$ can we replace the right-hand side by $C \sum_{n=0}^{+\infty} |a_n|^p$, with C not depending on $a_0, a_1, \ldots$, and $0 \leq C < 1$.

Reference

DE BRUIJN, N. G.: Problem 53. Wisk. Opgaven **19**, 107–108 (1950/54).

3.9.45 For $p = 2, 3, \ldots$ and $k = 1, 2, \ldots$ we have

$$\sum_{n=1}^{+\infty} \frac{1}{n^p (n+1)^p \cdots (n+k)^p} < \frac{1}{((k+1)!)^p} \left[1 + \frac{1}{(k+1)^p}\right].$$

Reference

MATSUOKA, Y.: Evaluation and estimation of series of certain special type. Sci. Rep. Kagoshima Univ. **13**, 11–15 (1964).

3.9.46 If $x \neq 0$, then

$$\frac{1}{x^2 + \frac{1}{2}} < \sum_{n=1}^{+\infty} \frac{2n}{(n^2 + x^2)^2} < \frac{1}{x^2}. \tag{1}$$

These inequalities are called MATHIEU's.

Proof. Let $n > 0$ and let x be real. Then

$$\frac{1}{\left(n - \frac{1}{2}\right)^2 + x^2 - \frac{1}{4}} - \frac{1}{\left(n + \frac{1}{2}\right)^2 + x^2 - \frac{1}{4}} = \frac{2n}{(n^2 + x^2 - n)(n^2 + x^2 + n)}$$

$$> \frac{2n}{(n^2 + x^2)^2}.$$

Summing for $n = 1, 2, \ldots$, we obtain, for $x \neq 0$,

$$\sum_{n=1}^{+\infty} \frac{2n}{(n^2 + x^2)^2} < \frac{1}{x^2}. \tag{2}$$

Let $n > 0$ and let x be real. Then

$$\frac{1}{\left(n - \frac{1}{2}\right)^2 + x^2 + \frac{1}{4}} - \frac{1}{\left(n + \frac{1}{2}\right)^2 + x^2 + \frac{1}{4}} = \frac{2n}{(n^2 + x^2)^2 + x^2 + \frac{1}{4}} < \frac{2n}{(n^2 + x^2)^2}.$$

Summing, for $n = 1, 2, \ldots$, we get

$$\frac{1}{x^2 + \frac{1}{2}} < \sum_{n=1}^{+\infty} \frac{2n}{(n^2 + x^2)^2}. \tag{3}$$

Inequalities (2) and (3) prove (1).

Remark. Inequality (2) was conjectured in 1890 by E. MATHIEU [1] and proved only in 1952 by L. BERG [2]. A very elegant and at the same time elementary proof of (1) given above is due to E. MAKAI [3].

M. TIDEMAN [4] has proved

$$\sum_{n=1}^{+\infty} \frac{ktn^{2k-1}}{(n^2 + t)^{k+1}} < \frac{1}{2} \qquad (t \geq 0;\ k = 1, 2, \ldots).$$

For $k = 1$ and $t = x^2$ one obtains inequality (2).

H. W. GOULD and T. A. CHAPMAN [5] have proved in 1962 the inequalities

$$\frac{1}{x^2 + 1} - \frac{1}{x^2 + (m + 1)^2} \leq \sum_{n=1}^{m} \frac{2n}{(n^2 + x^2)^2} \leq \frac{1}{x^2} - \frac{1}{x^2 + m^2},$$

from which we can derive weaker inequalities than (1).

See also papers [6]—[9] which are related to the MATHIEU inequalities.

References

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3.9.47 If the function f is defined by

$$xf(x) = \sum_{n=1}^{+\infty} \frac{n^{n-1}}{n!} (xe^{-x})^n \qquad (x > 1),$$

then

$$(-1)^k \frac{d^k f(x)}{dx^k} \geq 0 \quad \text{for} \quad k = 0, 1, 2, 3, 4.$$

This inequality does not hold for $k > 4$.

References

SHAH, S. M., and U. C. SHARMA: Some properties of a function of Ramanujan. J. Univ. Bombay **17**, 1—4 (1948).
BOAS, R. P.: A series considered by Ramanujan. Arch. Math. (Basel) **11**, 350—351 (1960).

3.9.48 Let f be a function defined over an interval (a, b). Suppose that f''' exists and that it is increasing on (a, b). Then, if $x \in (a+1, b-1)$, we have

$$f(x-1) - 2f(x) + f(x+1) > f''(x). \tag{1}$$

For instance, we have for $x > 1$,

$$(x-1)\log(x-1) - 2x\log x + (x+1)\log(x+1) > \frac{1}{x}. \tag{2}$$

Proof. By TAYLOR's theorem,

$$f(x-1) = f(x) - f'(x) + \frac{1}{2} f''(x) - \frac{1}{6} f'''(\xi),$$

$$f(x+1) = f(x) + f'(x) + \frac{1}{2} f''(x) + \frac{1}{6} f'''(\eta),$$

where ξ, η are some numbers satisfying

$$x - 1 < \xi < x < \eta < x + 1.$$

It follows that

$$f(x-1) - 2f(x) + f(x+1) = f''(x) + \frac{1}{6}[f'''(\eta) - f'''(\xi)].$$

Since f''' is a increasing function, we conclude that (1) is true. Inequality (2) follows from (1) if we take $f(x) = x\log x$. Then $x \mapsto f'''(x) = -1/x^2$ is an increasing function on $(0, +\infty)$.

3.9.49 Let f and g be real-valued functions defined over (a, b). Suppose that f and g are positive, continuous and differentiable on (a, b). Furthermore, suppose that f' and g' are positive on (a, b) and that $x \mapsto f'(x)/g'(x)$ is an increasing function on (a, b). Then:

1° $x \mapsto f(x)/g(x)$ is an increasing function on (a, b), or

2° $x \mapsto f(x)/g(x)$ is a decreasing function on (a, b), or

3° there is a number $c \in (a, b)$ such that $x \mapsto f(x)/g(x)$ is a decreasing function on (a, c) and an increasing function on (c, b).

Remark. This result is given without proof in: N. BOURBAKI: Fonctions d'une variable réelle. Paris 1958, Chap. 1, § 2, Exerc. 10.

3.9.50 Let f have a third derivative on $(0, 2a)$ with $f'''(t) \geq 0$, let $0 < x_k \leq a$ and $p_k > 0$ for $k = 1, \ldots, n$. Then

$$\text{(1)} \qquad \frac{\sum p_k f(x_k)}{\sum p_k} - f\left(\frac{\sum p_k x_k}{\sum p_k}\right) \leq \frac{\sum p_k f(2a - x_k)}{\sum p_k} - f\left(\frac{\sum p_k (2a - x_k)}{\sum p_k}\right),$$

where all the sums are over $k = 1, \ldots, n$. If $f'''(t) > 0$ on $(0, 2a)$, then equality occurs above if and only if all x_k are equal.

Remark. Inequality (1) is due to N. LEVINSON and it generalizes the following inequality of K. FAN

$$\frac{\prod_{k=1}^{n} x_k}{\left(\sum_{k=1}^{n} x_k\right)^n} \leq \frac{\prod_{k=1}^{n} (1 - x_k)}{\left(\sum_{k=1}^{n} (1 - x_k)\right)^n} \qquad \left(0 < x_k \leq \frac{1}{2}\right),$$

with equality if and only if all x_k are equal.

T. POPOVICIU gave a generalization of (1).

References

LEVINSON, N.: Generalization of an inequality of Ky Fan. J. Math. Anal. Appl. **8**, 133–134 (1964).

POPOVICIU, T.: Sur une inégalité de N. Levinson. Mathematica (Cluj) **6 (29)**, 301–306 (1964).

3.9.51 Let

$$\frac{1 + a_1 x + a_2 x^2 + \cdots}{1 + b_1 x + b_2 x^2 + \cdots} = 1 + c_1 x + c_2 x^2 + \cdots,$$

where $0 \leq a_i \leq 1$ and $0 \leq b_i \leq 1$ $(i = 1, 2, \ldots)$. Then c_n does not exceed the n-th FIBONACCI number.

Reference

SHAPIRO, H. S.: Problem E 1502. Amer. Math. Monthly **69**, 922–923 (1962).

3.9.52 Let $x \mapsto y(x)$, $x \mapsto w(x;t)$ and $x \mapsto g(x)$ be continuous functions of x in $[a, b]$. Let f have continuous, and w and g have sectionally continuous derivatives there (derivation always with respect to x). Assume that

$$\big(f(x)\, w'(x;t)\big)' - g(x)\, w(x;t) = 0 \quad \text{for} \quad a \leq x < t \quad \text{and} \quad t < x \leq b,$$

$$\lim_{\varepsilon \to 0^+} [w'(t-\varepsilon;t) - w'(t+\varepsilon;t)]\, f(t) = 1,$$

and

$$f(a)\, w'(a;t)\, y(a) = f(b)\, w'(b;t)\, y(b).$$

Then, we have

$$(1) \qquad |y(t)| \leq M(t) \left(\int_a^b \left(|f(x)^\beta y'(x)|^q + |g(x)^\delta y(x)| \right)^{\frac{\nu}{q}} dx \right)^{\frac{1}{\nu}},$$

where

$$M(t) = \left(\int_a^b \left(|f(x)^\alpha w'(x;t)|^p + |g(x)^\gamma w(x;t)|^\beta \right)^{\frac{\mu}{p}} dx \right)^{\frac{1}{\mu}},$$

and $\alpha + \beta = \gamma + \delta = p^{-1} + q^{-1} = \mu^{-1} + \nu^{-1}$ with $p \geq 1$ and $\mu \geq 1$.

A particularly interesting case of (1) is when $f(x) \geq 0$, $g(x) \geq 0$ and $\mu = p = q = \alpha^{-1} = \gamma^{-1}$. Then (1) takes the form

$$|y(t)|^2 \leq \tilde{M}(t)^2 \int_a^b \big(y'(x)^2 f(x) + y(x)^2 g(x)\big)\, dx,$$

where

$$\tilde{M}(t)^2 = w(t;t) + f(b)\, w'(b;t)\, w(b;t) - f(a)\, w'(a;t)\, w(a;t)$$

with $w'(b;t) = \dfrac{dw(x;t)}{dx}$ for $x = b$.

Reference

Block, H. D.: A class of inequalities. Proc. Amer. Math. Soc. **8**, 844–851 (1957).

3.9.53 Let F be a nonnegative convex and a nondecreasing function on $[a, +\infty)$ with an asymptote $y = x - \alpha$ $(\alpha > 0)$. Then the following inequalities hold:

$$F(x+h) \leq F(x) + h \quad \text{for} \quad x \geq a,\ h \geq 0;$$

$$F\big(F(2x)\big) \leq 2F(x) \quad \text{for} \quad x \geq a.$$

Reference

Pryce, J. D.: Problem 5266. Amer. Math. Monthly **73**, 212–213 (1966).

3.9.54 If f' is an increasing function, then

$$(1) \qquad f'(x+1) > f(x+1) - f(x) > f'(x).$$

Proof. By LAGRANGE's theorem,

$$f(x+1) - f(x) = f'(x+\theta) \qquad (0 < \theta < 1).$$

From this equality and the hypothesis for $f'(x)$, (1) follows.

Application. (1) may written in the form

$$f(x+1) - f(x) > f'(x) > f(x) - f(x-1). \tag{2}$$

Let $f(x) = \frac{2}{3} x^{3/2}$ and write (2) with x successively replaced by $1, \ldots, n$. Summing, we obtain

$$\frac{2}{3}\left((n+1)^{3/2} - 1\right) > \sum_{k=1}^{n} k^{1/2} > \frac{2}{3} n^{3/2}.$$

Revised from a proof by Z. POP-STOJANOVIĆ.

3.9.55 Let $a \leq b \leq c$. If f is continuous for $x \in [a, c]$ and if f' is increasing for $x \in (a, c)$, then

$$(b-a) f(c) + (c-b) f(a) \geq (c-a) f(b). \tag{1}$$

The inequality is reversed if f' is decreasing.

If f' is strictly monotone and a, b, c are all distinct, equality is excluded.

Proof. If f' is increasing, applying the mean value theorem, we obtain

$$\frac{f(c) - f(b)}{c - b} = f'(\beta) \geq f'(\alpha) = \frac{f(b) - f(a)}{b - a}$$

and (1) follows.

Other assertions are evident.

Example. Let $f(x) = \log(1+x)$. Derivative $f'(x) = \frac{1}{1+x}$ is decreasing for $x \geq 0$. Let

$$a = 0, \quad b = \frac{\alpha}{p}, \quad c = \frac{\alpha}{q} \qquad (\alpha > 0;\ 0 < q < p).$$

In this case, all conditions for applications of the inequality

$$(b-a) f(c) + (c-b) f(a) < (c-a) f(b)$$

are satisfied, and we get, for $\alpha > 0$ and $0 < q < p$,

$$\frac{\alpha}{p} \log\left(1 + \frac{\alpha}{q}\right) < \frac{\alpha}{q} \log\left(1 + \frac{\alpha}{p}\right),$$

i.e.,

$$\left(1 + \frac{\alpha}{q}\right)^q < \left(1 + \frac{\alpha}{p}\right)^p.$$

Reference

BUSH, K. A.: On an application of the mean value theorem. Amer. Math. Monthly **62**, 577–578 (1955).

Remark. Inequality (1) is, in fact, the definition of convex functions (see Theorem 1 of 1.4.3).

3.9.56 Let f be a real-valued function of a real variable, satisfying the following conditions:

$1°$ $f(0) = 0$,

$2°$ $f(x)$ is continuous for $x > 0$,

$3°$ $f(x)$ has the $(n-1)$-th derivative, and $x \mapsto (-1)^{n-1} f^{(n-1)}(x)$ is an increasing function for $x > 0$.

Then, for $x_i > 0$ $(i = 1, \ldots, n)$, we have

$$\sum (-1)^{k-1} \sum{}^{*} f(x_{i_1} + \cdots + x_{i_k}) > 0, \tag{1}$$

where Σ^* is the summation over all the combinations $i_1, \ldots, i_k$ of k natural numbers $1, \ldots, n$.

Proof. We shall prove (1) for the case $n = 3$. Inequality (1) can be proved analogously for arbitrary n. Consider the function

$$F(x, y, z) = f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(z + x) + f(x - y + z).$$

We shall prove that $F > 0$ for $x, y, z > 0$. Differentiating F with respect to x, and then y, we get

$$F'_x = f'(x) - f'(x + y) - f'(x + z) + f'(x + y + z),$$
$$F''_{xy} = -f''(x + y) + f''(x + y + z).$$

According to $3°$, we have $F''_{xy} > 0$. F'_x is, therefore, increasing with respect to y and vanishes for $y = 0$. Therefore, $F'_x > 0$. Again, we see that F is increasing with respect to x, and since it vanishes for $x = 0$, we get $F > 0$, which was to be proved.

Notice that if the n-th derivative exists, condition $3°$ can be replaced by $(-1)^{n-1} f^{(n)}(x) > 0$.

A number of interesting inequalities can be obtained from (1) by applying it to $f(x) = \log(x + 1)$, $f(x) = \log \dfrac{a^{x+1} - 1}{a - 1}$, and hence to the number-theoretical functions $\sigma(n)$, $\sigma_m(x)$, $\varphi(n)$. The results are:

If $x_1, \ldots, x_n$ are nonnegative real numbers, then

$$\prod_{k=1}^{n} \{\Pi^* (x_i + \cdots + x_{i_k} + 1)\}^{(-1)^{k-1}} \geq 1,$$

with equality if and only if at least one of the numbers x_i is zero.

If $a > 0$, and if $x_1, \ldots, x_n$ are nonnegative real numbers, then

$$\prod_{k=1}^{n} \left\{\Pi^* \frac{a^{x_{i_1} + \cdots + x_{i_k}} - 1}{a - 1}\right\}^{(-1)^{k-1}} \geq 1,$$

with equality if and only if at least one of the numbers x_i is zero.

$$\prod_{k=1}^{n} \{\Pi^* \sigma(a_{i_1} \cdots a_{i_k})\}^{(-1)^{k-1}} \geq 1,$$

with equality if and only if $a_1, \ldots, a_n$ are relatively prime.

$$\prod_{k=1}^{n} \{\Pi^* \sigma_m(a_{i_1} \cdots a_{i_k})\}^{(-1)^{k-1}} \geq 1,$$

with equality if and only if $a_1, \ldots, a_n$ are relatively prime.

For $n > 1$,

$$\prod_{k=1}^{n} \{\Pi^* \varphi(a_{i_1} \cdots a_{i_k})\}^{(-1)^{k-1}} \leq 1,$$

with equality if and only if $a_1, \ldots, a_n$ are relatively prime.

In the above inequalities Π^* is defined analogously to Σ^*.

The above interesting results are due to T. Popoviciu [1].

Remark. Inequality (1) was rediscovered by P. M. Vasić and it appears as one of his results in [2]. P. M. Vasić succeeded, however, in proving (1) for the case $n = 3$ without the supposition of differentiability, which he replaced by the weaker condition of convexity of order 2.

J. D. Kečkić has communicated to us that inequality (1) holds for a convex function of order $n - 1$ (see 1.4.3). In [3], he gave a proof of this statement for $n = 4$, indicating how it may be carried further for arbitrary n.

References

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3. Kečkić, J. D.: Some inequalities for convex functions of order n. Mat. Vesnik **7 (22)**, 67—73 (1970).

3.9.57 Let f be a twice differentiable function on $[0, a)$, and let it on that interval satisfy the following differential inequality

$$xf''(x) - (k-1) f'(x) \geq 0, \tag{1}$$

where k is a real number not equal to zero. Then

$$f\left(\sqrt[k]{\frac{x_1^k + \cdots + x_n^k}{n}}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n} \leq \frac{f\left(\sqrt[k]{x_1^k + \cdots + x_n^k}\right) + (n-1) f(0)}{n}, \tag{2}$$

for $x_i \in [0, a)$ $(i = 1, \ldots, n)$ and $\sqrt[k]{x_1^k + \cdots + x_n^k} \in [0, a)$.

Furthermore, if f is $k+1$ times differentiable on $[0, a)$, and if it satisfies (1), then

$$(3)\quad f\left(\sqrt[k]{\frac{x_1^k + \cdots + x_n^k}{n}}\right) + \sum_{i=1}^{k-1} \frac{f^{(i)}(0)}{i!}\left(M_i^i - M_k^i\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n}$$

$$\le \frac{f\left(\sqrt[k]{x_1^k + \cdots + x_n^k}\right) + (n-1) f(0) + \sum_{i=1}^{k-1} \frac{f^{(i)}(0)}{i!}\left(s_i^i - s_k^i\right)}{n},$$

where $s_i = \sqrt[i]{x_1^i + \cdots + x_n^i}$, $M_i = \sqrt[i]{\frac{x_1^i + \cdots + x_n^i}{n}}$, and $x_i \in [0, a)$ $(i = 1, \ldots, n)$, and $s_k \in [0, a)$.

The above results are due to J. D. Kečkić and I. B. Lacković [1]. The same inequalities were first proved by P. M. Vasić [2] who supposed that f has the $(k+1)$-th derivative on $[0, a)$, that it is convex of order k (see 1.4) and that for (2) $f'(0) = \cdots = f^{(k-1)}(0) = 0$, while for (3) that $f'(0), \ldots, f^{(k-1)}(0)$ are arbitrary.

Those inequalities of P. M. Vasić appear as generalizations of certain inequalities of D. Marković [3].

Further generalizations of inequalities (2) and (3) are also given in [1].

References

1. Kečkić, J. D., and I. B. Lacković: Generalizations of some inequalities of P. M. Vasić. Mat. Vesnik **7 (22)**, 74–78 (1970).
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3. Marković, D.: On an inequality of M. Petrović (Serbian). Bull. Soc. Math. Phys. Serbie **11**, 45–53 (1959).

3.9.58 Let f be a convex function on $[0, a)$ and let $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$ be two sequences of nonnegative real numbers such that $p_k \ge 1$ for $k = 1, \ldots, n$. If

$$\sum_{k=1}^{n_1-1} p_k x_k = \sum_{k=n_1}^{n_2-1} p_k x_k = \cdots = \sum_{k=n_{r-1}}^{n} p_k x_k,$$

then

$$(1)\qquad \sum_{k=1}^{n} p_k f(x_k) \le r f\left(\frac{1}{r} \sum_{k=1}^{n} p_k x_k\right) - \left(r - \sum_{k=1}^{n} p_k\right) f(0),$$

where $x_k \in [0, a)$, for $k = 1, \ldots, n$ and $\frac{1}{r} \sum_{k=1}^{n} p_k x_k \in [0, a)$.

Remark. This result is due to P. M. Vasić [1]. For $p_k = 1$ $(k = 1, \ldots, n)$ one obtains an inequality of A. W. Marshall, I. Olkin and F. Proschan [2], which includes, as a special case, the following inequality

$$(2) \qquad \frac{1}{n} \sum_{k=1}^{n} |a_k - a|^s \leq \binom{n}{2}^{s-1} \left(\sum_{k=1}^{n} |a_k - a| \right)^s,$$

where $a = \frac{1}{n} \sum_{k=1}^{n} a_k$, $a_k \geq 0$ and $s \geq 1$.

Inequality (2) appears in Statistics (consult the papers [3] of S. Gatti, [4] of Z. W. Birnbaum, and [5] of M. de Novellis).

References

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2. Marshall, A. W., I. Olkin and F. Proschan: Monotonicity of ratios of means and other applications of majorization. In: Inequalities, edited by O. Shisha. New York-London 1967, pp. 177—190.
3. Gatti, S.: Sul massimo di un indice di anormalità. Metron **18**, 181—188 (1956).
4. Birnbaum, Z. W.: On an inequality due to S. Gatti. Metron **19**, 243—244 (1958).
5. de Novellis, M.: Some applications and developments of Gatti-Birnbaum inequality. Metron **19**, 245—247 (1958).

3.9.59 A sequence a_n $(n = 0, 1, \ldots)$ is called logarithmically concave if

$$a_n^2 - a_{n-1} a_{n+1} \geq 0 \qquad (n = 1, 2, \ldots).$$

If the sequences a_n, b_n are positive and logarithmically concave, then their convolution

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

has the same properties.

Reference

Lorentz, G. G.: Problem 4517. Amer. Math. Monthly **61**, 205 (1954).

3.9.60 If $f(x_1, \ldots, x_n)$ is a concave, symmetric and homogeneous function of the first degree, then

$$f(x_1, \ldots, x_n) \leq \frac{x_1 + \cdots + x_n}{n} f(1, \ldots, 1).$$

This result is due to A. E. Gel'man.

3.9.61 Let $x_{\mu\nu}$ $(\mu = 1, \ldots, m$ and $\nu = 1, \ldots, n)$, p and r be positive numbers, with $\sum_{\mu=1}^{m} x_{\mu\nu} = c_\nu$. Then

$$(1) \qquad \sum_{\mu=1}^{m} \left(\sum_{\nu=1}^{n} x_{\mu\nu}^r \right)^p \leq n^{1-pr} \left(\sum_{\nu=1}^{n} c_\nu^r \right)$$

if $r \leq 1$ and $pr \leq 1$. The converse inequality holds if $r \geq 1$ and $pr \geq 1$.

MINKOWSKI's inequality results for $p = 1/r$.

The integral analogue of (1) is given by:

Let f be a positive function almost everywhere, summable in the rectangle $a_1 < x < a_2$, $b_1 < y < b_2$, such that $\int_{a_1}^{a_2} f(x, y)\, dx = C(y)$, where $C(y) \in L(b_1, b_2)$ is a known function. Then

$$\int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} f(x, y)^r dy \right)^p dx \leq (a_2 - a_1)^{1-pr} \left(\int_{b_1}^{b_2} C(y)^r dy \right)^p$$

if $r \leq 1$ and $pr \leq 1$. The converse inequality holds if $r \geq 1$ and $pr \geq 1$.

Reference

MIKOLÁS, M.: Sur un problème d'extremum et une extension de l'inégalité de Minkowski. Ann. Univ. Sci. Budapest Eötvös, Sect. Math. **1**, **101**—**106** (1958).

3.9.62 Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers not all equal to zero, and let $\sum_{k=1}^{+\infty} k^2 a_k^2 < +\infty$. Then

$$\left(\sum_{k=1}^{+\infty} a_k \right)^4 \leq \pi^2 \left(\sum_{k=1}^{+\infty} a_k^2 \right) \left(\sum_{k=1}^{+\infty} k^2 a_k^2 \right), \tag{1}$$

where π^2 is the best possible constant.

The above inequality has the following integral analogue: let f be a nonnegative real-valued function on $[0, +\infty)$, such that $x \mapsto f(x)^2$ and $x \mapsto x^2 f(x)^2$ are integrable functions on the same interval. Then

$$\left(\int_0^{+\infty} f(x)\, dx \right)^4 \leq \pi^2 \left(\int_0^{+\infty} f(x)^2 dx \right) \left(\int_0^{+\infty} x^2 f(x)^2\, dx \right), \tag{2}$$

where π^2 is also the best possible constant.

Without any difficulty, from (1) we can pass to (2), and conversely. Inequalities (1) and (2) are called CARLSON's [1].

Generalizations. We note many extensions of (1) and (2). B. SZ.-NAGY [2] demonstrated the following inequalities

$$\left| \sum_{k=1}^{+\infty} a_k \right|^q + \left| \sum_{k=1}^{+\infty} (-1)^k a_k \right|^q < \frac{\pi q}{2} \left(\sum_{k=1}^{+\infty} k^{p_1} |a_k|^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{k=1}^{+\infty} |a_k|^{p_2} \right)^{\frac{1}{(p_2-1)p_1}}$$

and

$$\left| \sum_{k=1}^{+\infty} a_k \right|^q < \frac{\pi q}{2} \left(\sum_{k=1}^{+\infty} \left(k - \frac{1}{2}\right)^{p_1} |a_k|^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{k=1}^{+\infty} |a_k|^{p_2} \right)^{\frac{1}{(p_2-1)p_1}},$$

where $a_1, a_2, \ldots$ are real numbers, $q = 1 + \frac{p_2}{p_1(p_2 - 1)}$, $1 \leq p_1 \leq 2$ and $1 < p_2 \leq 2$.

We assume that the series in question converge.

V. I. LEVIN [3], improving the results of W. B. CATON [4] and R. BELLMAN [5], showed that, if $p > 1$, $q > 1$, $\lambda > 0$, $\mu > 0$ and if f is a real-valued nonnegative function on $[0, +\infty)$, such that $x \mapsto x^{p-1-\lambda} f(x)^p$ and $x \mapsto x^{q-1+\mu} f(x)^q$ are integrable functions on the considered interval, then

$$\int_0^{+\infty} f(x)\, dx \leq C \left(\int_0^{+\infty} x^{p-1-\lambda} f(x)^p dx \right)^{\frac{\mu}{p\mu + q\lambda}} \left(\int_0^{+\infty} x^{q-1+\mu} f(x)^q dx \right)^{\frac{\lambda}{p\mu + q\lambda}}$$

with

$$C = \left(\frac{1}{ps}\right)^s \left(\frac{1}{qt}\right)^t \left(\frac{\Gamma\left(\frac{s}{1-s-t}\right) \Gamma\left(\frac{t}{1-s-t}\right)}{(\lambda + \mu)\, \Gamma\left(\frac{s+t}{1-s-t}\right)} \right)^{1-s-t},$$

$$s = \frac{\mu}{p\mu + q\lambda}, \quad t = \frac{\lambda}{p\mu + q\lambda}.$$

Constant C is the best possible.

The problem of determining the best possible constant K in a more general inequality

$$\int_0^{+\infty} x^{\alpha} |f(x)|^{\beta} dx \leq K \left(\int_0^{+\infty} x^{\alpha_1} |f(x)|^{\beta_1} dx \right)^{a_1} \left(\int_0^{+\infty} x^{\alpha_2} |f(x)|^{\beta_2} dx \right)^{a_2},$$

where the exponents are real numbers subject to certain conditions, was considered in a number of papers. The results obtained by B. KJELLBERG (see [6], [7] and [8]) are of special interest. In his papers extensive literature on that subject can also be found.

Concerning other generalizations, consult [9]—[11].

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3.9.63 Let (a_{ij}) be an $m\times n$ matrix of nonnegative terms. Then

$$mn\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\sum_{r=1}^{m}a_{rj}\sum_{s=1}^{n}a_{is}\geq\left(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\right)^3, \tag{1}$$

with equality if and only if all the row sums are equal, or all the column sums are equal, or both.

The above inequality is due to F. V. ATKINSON, G. A. WATTERSON and P. A. P. MORAN [1]. They also used (1) to obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}p_iq_j\sum_{r=1}^{m}a_{rj}p_r\sum_{s=1}^{n}a_{is}q_s\geq\left(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}p_iq_j\right)^3, \tag{2}$$

where $p_i\geq 0,\quad q_i\geq 0,\quad \sum_{i=1}^{m}p_i=\sum_{i=1}^{n}q_i=1.$

Putting in (2) $m=n$, $a_{ij}=a_{ji}$, and $p_i=q_i$ an inequality conjectured by S. P. H. MANDEL and I. M. HUGHES [2] on genetical grounds is obtained.

There is an obvious integral analogue of (1). If $K(x,y)\geq 0$ is integrable on the rectangle $0\leq x\leq a$, $0\leq y\leq b$, then

$$ab\int_0^a\int_0^b\int_0^a\int_0^b K(x,t)\,K(x,y)\,K(s,y)\,dx\,dy\,ds\,dt\geq\left(\int_0^a\int_0^b K(x,y)\,dx\,dy\right)^3. \tag{3}$$

If $a=b$ and $K(x,y)=K(y,x)$, inequality (3) can be written in the form

$$a^2\int_0^a\int_0^a K_3(x,y)\,dx\,dy\geq\left(\int_0^a\int_0^a K(x,y)\,dx\,dy\right)^3, \tag{4}$$

where K_3 is the third-order iterate of K in the sense of the theory of iterated kernels of integral equations.

Inequality (4) suggests a more general result

$$a^{n-1}\int_0^a\int_0^a K_n(x,y)\,dx\,dy\geq\left(\int_0^a\int_0^a K(x,y)\,dx\,dy\right)^n. \tag{5}$$

This inequality was proved in [1] for all n of the form 2^r3^s, and was conjectured for other positive integral values of n.

J. F. C. KINGMAN [3] gave a more direct proof of (2) which is based on the convexity of $f(x)=x^k$ for $k\geq 1$.

A generalization to products of sums of $a_{ijk\ldots}$ taken over subsets of the index set $\{i, j, k, \ldots\}$ was also obtained in [3]. For example,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} a_{ijk}p_i q_j r_k \sum_{s=1}^{n}\sum_{t=1}^{l} a_{ist} q_s r_t \sum_{s=1}^{m}\sum_{t=1}^{l} a_{sjt} p_s r_t \sum_{s=1}^{m}\sum_{t=1}^{n} a_{stk} p_s q_t$$

$$\geq \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} a_{ijk} p_i q_j r_k\right)^4 .$$

for $p_i \geq 0$, $q_i \geq 0$, $r_i \geq 0$, and $\sum_{i=1}^{m} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{l} r_i = 1$.

Noting that the notion of a "partial average" is a particular example of that of a conditional expectation, and that a conditional expectation is a special sort of Radon-Nikodým derivative, J. F. C. Kingman established an inequality involving such derivatives, which is a generalization of (2). For this generalization consult [4].

Using Kingman's method of proof, P. R. Beesack [5] proved the somewhat weaker inequality than (5), namely

$$\int_0^a\int_0^a K_n(x, y)\,dx\,dy \geq a^{n+1} \exp\left(\frac{n\int_0^a\int_0^a \log K(x, y)\,dx\,dy}{a^2}\right), \tag{6}$$

which, however, holds for all $n \geq 1$ and for an arbitrary nonnegative kernel K. In case $K = \text{const} > 0$, both (5) and (6) are true with equality holding, but in general (5) is better than (6).

Using a corresponding result for symmetric matrices with nonnegative elements due to H. P. Mulholland and C. A. B. Smith [6], P. R. Beesack in [5] showed that the conjecture (5) is true for all $n \geq 1$ and nonnegative symmetric kernels K. More generally it was shown that for such kernels

$$\left(\int_0^a v(x)^2 dx\right)^{n-1} \int_0^a\int_0^a v(x)\, v(y)\, K_n(x, y)\, dx\, dy$$

$$\geq \left(\int_0^a\int_0^a v(x)\, v(y)\, K(x, y)\, dx\, dy\right)^n$$

holds for arbitrary nonnegative continuous functions v; setting $v(x) \equiv 1$, (5) is obtained.

In fact, in Beesack's paper [5] inequality (6) was formulated in a more general form. In addition, the same technique was used to obtain a lower bound for the convolution of n positive functions, which reads:

If $f_1, \ldots, f_n$ are nonnegative LEBESGUE square integrable functions on $[0, a)$ for all $a > 0$, then for all $n \geq 2$, and $x \geq 0$,

$$f_1 * \cdots * f_n(x) \geq \frac{x^{n-1}}{(n-1)!} \exp\left\{(n-1)\, x^{-n+1} \int_0^x (x-u)^{n-2} \sum_{j=1}^{n} \log f_j(u)\, du\right\},$$

where $f_i * f_j(x)$ denotes the convolution $\int_0^x f_i(t)\, f_j(x-t)\, dt$.

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3.9.64 Let u and h be continuous and nonnegative functions on $[0, 1]$. Let $c \geq 0$ be a constant. If, for $0 \leq x \leq 1$,

$$u(x) \leq c + \int_0^x h(t)\, u(t)\, dt, \tag{1}$$

then, for $0 \leq x \leq 1$,

$$u(x) \leq c \exp\left(\int_0^x h(t)\, dt\right). \tag{2}$$

This result is due to T. H. GRONWALL [1].

Various linear generalizations of (2) have been given. See [2]–[6].

We present below a generalization of the GRONWALL inequality which is due to S. C. CHU and F. T. METCALF [2], namely:

Let u and f be real continuous functions on $[0, 1]$. Let K be continuous and nonnegative on the triangle $0 \leq y \leq x \leq 1$. If

$$u(x) \leq f(x) + \int_0^x K(x, y)\, u(y)\, dy \quad \text{for} \quad 0 \leq x \leq 1,$$

then

$$u(x) \leq f(x) + \int_0^x H(x, y)\, f(y)\, dy \quad \text{for} \quad 0 \leq x \leq 1,$$

where, for $0 \leq y \leq x \leq 1$,

$$H(x, y) = \sum_{i=1}^{+\infty} K_i(x, y)$$

is the resolvent kernel and the K_i ($i = 1, 2, \ldots$) are the iterated kernels of K.

There are other generalizations (as well as applications) of GRONWALL's inequality: some nonlinear generalizations are given by D. WILLETT and J. S. W. WONG [7]; a slight sharpening of the result of D. C. CHU and F. T. METCALF as well as some applications to VOLTERRA integral equations is given by P. R. BEESACK [8].

See also paper [9] of W. W. SCHMAEDEKE and G. R. SELL.

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3.9.65 Consider the homogeneous linear differential equation

$$x'' + g(t)\, x' + f(t)\, x = 0, \tag{1}$$

whose coefficients are real and continuous functions. Let x be a nontrivial integral of (1) such that

$$x(0) = 0 \quad \text{and} \quad x(h) = 0 \quad (h > 0).$$

Applying a general theorem due to C. DE LA VALLÉE POUSSIN [1] to (1), we obtain the following inequality

$$1 < 2mh + \frac{1}{2} kh^2, \tag{2}$$

where $2m = \max\limits_{0 \le t \le h} |g(t)|$ and $k = \max\limits_{0 \le t \le h} |f(t)|$.

PH. HARTMAN and A. WINTNER [2] improved inequality (2) and proved that

$$1 < mh + \frac{1}{6} kh^2.$$

The best possible inequality of this type was obtained by Z. OPIAL [3] who proved that

$$\pi^2 \le 4mh + kh^2, \tag{3}$$

where equality can hold only in the case $m = 0$.

Though the proof of (3) is elementary, it is rather long, and so, following Z. OPIAL, we give a proof of the somewhat weaker inequality

$$\pi^2 \le 2\pi mh + kh^2. \tag{4}$$

Let x be a nontrivial integral of (1) such that $x(0) = x(h) = 0$. Then

$$x''(t) + g(t)\, x'(t) + f(t)\, x(t) \equiv 0.$$

Multiplying this identity by $x(t)$ and integrating from 0 to h, we get

$$\int_0^h x''(t)\, x(t)\, dt + \int_0^h g(t)\, x'(t)\, x(t)\, dt + \int_0^h f(t)\, x(t)^2\, dt = 0.$$

However,

$$\int_0^h x''(t)\, x(t)\, dt = x'(t)\, x(t)\Big|_0^h - \int_0^h x'(t)^2\, dt = - \int_0^h x'(t)^2\, dt.$$

Therefore,

$$- \int_0^h x'(t)^2\, dt + 2m \int_0^h |x(t)| \cdot |x'(t)|\, dt + k \int_0^h x(t)^2\, dt \ge 0, \tag{5}$$

where equality is possible only when $m = 0$. Using the inequality (see 2.23.2)

$$\int_0^h x(t)^2\, dt \le \frac{h^2}{\pi^2} \int_0^h x'(t)^2\, dt, \tag{6}$$

and applying the BUNIAKOWSKI-SCHWARZ inequality, we get

$$\int_0^h |x(t)|\, |x'(t)|\, dt \le \left(\int_0^h x'(t)^2\, dt \cdot \int_0^h x(t)^2\, dt \right)^{1/2} \le \frac{h}{\pi} \int_0^h x'(t)^2\, dt. \tag{7}$$

Inequalities (4), (5) and (6) together yield

$$-\int_0^h x'(t)^2\,dt + 2m\frac{h}{\pi}\int_0^h x'(t)^2\,dt + k\frac{h^2}{\pi^2}\int_0^h x'(t)^2\,dt \geq 0,$$

which implies (4).

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2. HARTMAN, PH., and A. WINTNER: On an oscillation criterion of de la Vallée Poussin. Quart. Appl. Math. **13**, 330–332 (1955).
3. OPIAL, Z.: Sur une inégalité de C. de la Vallée Poussin. Ann. Polon. Math. **6**, 87–91 (1959).

3.9.66 Let p be a real continuous function of the real variable x on $[a, b]$. If the differential equation

$$y'' + p(x)\,y = 0$$

has a nontrivial solution y that vanishes at two points of $[a, b]$, then, according to A. M. LYAPUNOV, p is subject to the inequality

$$(1) \qquad (b-a)\int_a^b |p(x)|\,dx > 4.$$

This inequality is sharp in the sense that the constant 4 cannot be replaced by a large number.

Z. NEHARI claimed that in [1] he has proved the following statement:

Let $p_1, \ldots, p_n$ be real continuous functions in $[a, b]$. If the differential equation

$$y^{(n)} + p_n(x)\,y^{(n-1)} + \cdots + p_1(x)\,y = 0$$

has a nontrivial solution y that has n zeros in $[a, b]$, then

$$(2) \qquad \sum_{k=1}^{n} 2^k (b-a)^{n-k} \int_a^b |p_k(x)|\,dx > 2^{n+1}.$$

As we can see from [2], Z. NEHARI himself communicated that the above inequality is undecided since the argument given in [1] is invalid.

A. M. FINK and D. F. ST. MARY demonstrated that (2) is correct for $n = 2$. In fact, they proved a stronger result for

$$(3) \qquad y'' + g(x)\,y' + f(x)\,y = 0,$$

which reads:

Let a and b be successive zeros of a nontrivial solution to (3) where f and g are integrable functions. Then

$$(4) \qquad (b-a)\int_a^b f^+(x)\,dx - 4\exp\left(-\frac{1}{2}\int_a^b |g(x)|\,dx\right) > 0$$

and, a fortiori,

$$(b-a)\int_a^b f^+(x)\,dx + 2\int_a^b |g(x)|\,dx > 4,$$

where $f^+(x) = \max(0, f(x))$.

Inequality (4) is sharp since it reduces to LYAPUNOV's inequality (1), when $g(x) \equiv 0$, and this is known to be sharp (see, for example, [3]).

H. HOCHSTADT [4] obtained the following result:

Let p and q be integrable functions in $[a, b]$. If

$$y^{(n)} - p(x)\,y^{(n-1)} - q(x)\,y = 0 \qquad (n \geq 2)$$

has a nontrivial solution y that has at least n zeros in $[a, b]$, then

$$\left((b-a)^{n-1}\int_a^b |q(x)|\,dx\right)^{1/n} + \frac{1}{n}\int_a^b |p(x)|\,dx \geq 2.$$

In connection with the more general results about this problem, see paper [5] of P. HARTMAN which also contains six bibliographical items.

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3.9.67 Let V be a vector space over the field of real numbers, and let the operator $\|x\|$ be defined for all $x \in V$, satisfying the following conditions

(1) $$\|x\| \quad \text{is a nonnegative real number},$$

(2) $$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2.$$

Then

$$\|x+y\| \leq \|x\| + \|y\|.$$

Proof. We first observe that $\|0\| = 0$ and $\|-x\| = \|x\|$. Define an inner product (x, y) by

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

We have immediately $(x, x) = \|x\|^2$ and $(x, y) = (y, x)$. By a repeated use of the relation (2) we obtain $(x + z, y) = (x, y) + (z, y)$. It then follows that $(rx, y) = r(x, y)$ for all rational numbers r. Hence, for all rational numbers r we have

$$0 \leq (rx + y, rx + y) = r(x, x)^2 + 2r(x, y) + (y, y),$$

which implies the CAUCHY inequality

$$|(x, y)| \leq \|x\| \, \|y\|.$$

However,

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + 2(x, y) + (y, y) \\ &\leq \|x\|^2 + 2\,|(x, y)| + \|y\|^2 \\ &\leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \end{aligned}$$

i.e., $\|x + y\| \leq \|x\| + \|y\|$.

Reference

KNUTH, D. E., W. G. DOTSON and G. C. MARSHALL: Problem 5264. Amer. Math. Monthly **73**, 211–212 (1966).

3.9.68 If α and β are vectors in a BANACH space and $p \geq 1$, then

$$2^{-p}\,|\alpha - \beta|^p \leq \big||\alpha|^{p-1}\,\alpha - |\beta|^{p-1}\,\beta\big| \leq 3p\,|\alpha - \beta|\,(|\alpha| + |\beta|)^{p-1}.$$

Remark. The above result is given in [1] without proof.

Reference

1. BOURBAKI, N.: Intégration. Paris 1965, Chap. 4, § 6, Exerc. 10, p. 257.

3.9.69 Let the arithmetic and geometric means of the real nonnegative numbers $a_1, \ldots, a_n$ taken $n - 1$ at a time be denoted by

$$\alpha_i = \frac{a_1 + \cdots + a_n - a_i}{n - 1}, \qquad \gamma_i = \left(\frac{a_1 \cdots a_n}{a_i}\right)^{1/(n-1)} \qquad (i = 1, \ldots, n).$$

Then

$$\frac{\gamma_1 + \cdots + \gamma_n}{n} \leq (\alpha_1 \cdots \alpha_n)^{1/n} \qquad (n = 3, 4, \ldots).$$

Reference

CARLSON, B. C.: Inequality of mixed arithmetic and geometric means. SIAM Review (to appear).

3.9.70 If $a_i \geq 0$, $n > 2$ and not all the members of the sequence $a = (a_1, \ldots, a_n)$ are equal, the following inequality holds

$$(1) \qquad (n - 2) \sum_{i=1}^{n} a_i + n(a_1 \cdots a_n)^{1/n} - 2 \sum_{1 \leq i < j \leq} (a_n^i a_j)^{1/2} \geq 0,$$

with equality if and only if, for some i,

$$a_i = 0, \quad a_1 = \cdots = a_{i-1} = a_{i+1} = \cdots = a_n.$$

Proof. Consider the function

$$g_n(a) = (n-2)\sum_{i=1} a_i + n(a_1 \cdots a_n)^{1/n} - 2 \sum_{1 \le i < j \le n} (a_i a_j)^{1/2}$$

and suppose, without loss of generality, that $a_1 \le a_2 \le \cdots \le a_n$.

Suppose now that (1) holds, for some $n > 2$, i.e., that $g_n(a) > 0$, and put $a_1' = a_1$ and $a_i' = A = (a_2 \cdots a_{n+1})^{1/n}$ $(i = 2, \ldots, n+1)$. Then

$$g_{n+1}(a') = (n-1)\, a_1 + (n+1)\, a_1^{\frac{1}{n+1}} A^{\frac{n}{n+1}} - 2n(a_1 A)^{1/2}.$$

If $a_1 = 0$, then $g_{n+1}(a') = 0$. Let now $a_1 > 0$. Then

$$\frac{g_{n+1}(a')}{2n a_1^{1/(n+1)}} = \frac{n-1}{2n} a_1^{\frac{n}{n+1}} + \frac{n+1}{2n} A^{\frac{n}{n+1}} - a_1^{\frac{n-1}{2(n+1)}} A^{1/2}.$$

Applying the arithmetic-geometric mean inequality we have $g_{n+1}(a') \ge 0$, with equality if and only if $a_1 = A$, i.e., $a_1 = \cdots = a_{n+1}$, and this case was excluded from our consideration.

Therefore $g_{n+1}(a') \ge 0$, equality holding if and only if $a_1 = 0$.

Consider now the difference $g_{n+1}(a) - g_{n+1}(a')$. We have

$$\begin{aligned} g_{n+1}(a) &- g_{n+1}(a') \\ &= (n-1)\sum_{i=2}^{n+1} a_i - 2a_1^{1/2}\left(\sum_{i=2}^{n+1} a_i^{1/2} - nA^{1/2}\right) - 2\sum_{2 \le i < j \le n+1} (a_i a_j)^{1/2} \\ &= g_n(\bar{a}) + \sum_{i=2}^{n+1} a_i - 2a_1^{1/2}\left(\sum_{i=2}^{n+1} a_i^{1/2} - nA^{1/2}\right) - n\prod_{i=2}^{n+1} a_i^{1/n}, \end{aligned}$$

where $\bar{a} = (a_2, \ldots, a_{n+1})$. Since $a_1 \le A$, $(a_2 \cdots a_{n+1})^{1/n} = A$ and $g_n(\bar{a}) \ge 0$, according to the hypothesis, and since the term in bracket is ≥ 0, we have

$$\begin{aligned} g_{n+1}(a) - g_{n+1}(a') &\ge \sum_{i=2}^{n+1} a_i - 2A^{1/2}\sum_{i=2}^{n+1} a_i^{1/2} + nA \\ &= \sum_{i=2}^{n+1} (a_i^{1/2} - A^{1/2})^2 \\ &\ge 0, \end{aligned}$$

with equality if and only if $a_i = A$ $(i = 2, \ldots, n+1)$, i.e., $a_2 = \cdots = a_{n+1}$.

This proves inequality (1).

Reference

Kober, H.: On the arithmetic and geometric means and on Hölder's inequality. Proc. Amer. Math. Soc. 9, 452–459 (1958).

3.9.71 Let f be a real function which on an interval whose length is not less than 2 satisfies the conditions $|f(x)| \leq 1$ and $|f''(x)| \leq 1$. Then

$$(1) \qquad |f'(x)| \leq 2,$$

where the constant 2 is the best possible.

Proof. Without any loss of generality we can suppose that $0 \leq x \leq 2$. Then

$$f(x) - f(0) = xf'(x) - \frac{1}{2} x^2 f''(t_1) \quad \text{for} \quad 0 \leq t_1 \leq x \leq 2,$$

$$f(2) - f(x) = (2 - x) f'(x) + \frac{1}{2} (2 - x)^2 f''(t_2) \quad \text{for} \quad 0 \leq x \leq t_2 \leq 2,$$

and, therefore,

$$f(2) - f(0) = 2f'(x) - \frac{1}{2} x^2 f''(t_1) + \frac{1}{2} (2 - x)^2 f''(t_2),$$

i.e.,

$$2|f'(x)| \leq 1 + 1 + \frac{1}{2} x^2 + \frac{1}{2} (2 - x)^2 = 4 - x(2 - x) \leq 4,$$

or

$$|f'(x)| \leq 2.$$

The function f defined by $f(x) = \frac{1}{2} x^2 - 1$ shows that the sign of equality can actually hold in (1).

The above result is due to E. Landau [1]. It was generalized by V. G. Avakumović and S. Aljančić [2] who proved by geometric arguments that if f is a real function such that $|f''(x)| \leq 1$ for $0 \leq x \leq 1$, then, for $0 \leq x \leq 1$,

$$|f'(x) - f(1) + f(0)| \leq \frac{1}{2} - x + x^2,$$

and that this bound cannot be improved.

They also proved that the following conditions

$$|f''(x)| \leq 1 \quad \text{for} \quad 0 \leq x \leq 1, \quad \text{and} \quad f'(0) = f'(1)$$

imply

$$|f'(x) - f(1) + f(0)| \leq \frac{1}{4} \quad \text{for} \quad 0 \leq x \leq 1.$$

This bound cannot be improved.

Remark. The above result of E. Landau, transcribed for the interval [0, 1], reads: If $|f''(x)| \leq 1$ and $|f(x)| \leq 1/4$ for $0 \leq x \leq 1$, then $|f'(x)| \leq 1$ for $0 \leq x \leq 1$.

Comment by J. D. Kečkić. Using the same method of proof as Landau's, the following result can be proved:

Let f be a real function which on an interval whose length is not less than a satisfies the conditions $|f(x)| \leq 1$ and $|f''(x)| \leq 1$. Then

$$|f'(x)| \leq \frac{2}{a} + \frac{a}{2}.$$

References

1. LANDAU, E.: Einige Ungleichungen für zweimal differentierbare Funktionen. Proc. London Math. Soc. **(2) 13**, 43–49 (1914).
2. AVAKUMOVIĆ, V. G., and S. ALJANČIĆ: Sur la meilleure limite de la dérivée d'une fonction assujettie à des conditions supplémentaires. Acad. Serbe Sci. Publ. Inst. Math. **3**, 235–242 (1950).

3.9.72 If f is a real periodic function with a period $2T$ and if p and q are positive integers, then

$$\max |f^{(p)}(t)| \leq 2T\left(\frac{T}{\pi}\right)^q \max |f^{(p+q+1)}(t)|.$$

Reference

PICONE, M.: Relazioni fra le derivate delle funzioni periodiche. Boll. Un. Mat. Ital. **6**, 251–253 (1927).

3.9.73 If $p_i > 0$, $q_i > 0$ $(i = 1, \ldots, n)$ and if

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i, \tag{1}$$

then

$$\sum_{i=1}^{n} p_i \log p_i \geq \sum_{i=1}^{n} p_i \log q_i. \tag{2}$$

Proof. Starting with inequality $x \log x \geq x - 1$, which holds for $x > 0$, we have

$$\frac{p_i}{q_i} \log \frac{p_i}{q_i} \geq \frac{p_i}{q_i} - 1.$$

Since $q_i > 0$, the last inequality yields

$$p_i \log \frac{p_i}{q_i} \geq p_i - q_i. \tag{3}$$

Adding inequalities (3) for $i = 1, \ldots, n$ and taking (1) into account, we get inequality (2).

Equality holds in (2) if and only if $p_i = q_i$ $(i = 1, \ldots, n)$.

Remark 1. Inequality (1) appears in the theory of information. See, for example, [1] and [2].

Remark 2. J. ACZÉL (see [3]) proved, in connection with a problem of J. PFANZAGL, that

$$\sum_{i=1}^{n} p_i f(p_i) \geq \sum_{i=1}^{n} p_i f(q_i) \text{ with } \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1,\ p_i > 0,\ q_i > 0 \quad (i = 1, \ldots, n > 2)$$

is equivalent to

$$f(p) = a \log p + b \qquad (a > 0),$$

if f is a differentiable function.

P. FISCHER [4] proved that all the solutions of inequality

$$\sum_{i=1}^{n} p_i f(p_i) \geq \sum_{i=1}^{n} p_i f(q_i) \qquad (n \geq 2) \tag{4}$$

with the cited conditions, are monotone, while for $n \geq 3$ they are differentiable.

GY. MUSZÉLY (see [5]) determined for $n = 2$ all the solutions of inequality (4) which are continuous on $(0, 1)$.

References

1. BRILLOUIN, L.: Science and Information Theory. New York **1956**, pp. 13–14.
2. KIRMSER, P. G.: Problem E 1274. Amer. Math. Monthly **64**, 432 (1957).
3. ACZÉL, J.: Problem 3 from Reports of Meetings. Aequationes Math. **1**, 300 (1968).
4. FISCHER, P.: Sur l'inégalité $\sum_{i=1}^{n} p_i f(p_i) \geq \sum_{i=1}^{n} p_i f(q_i)$. Aequationes Math. **2**, 363 (1969).
5. MUSZÉLY, GY.: Über die stetigen Lösungen der Ungleichung $pf(p) + (1 - p) \times f(1 - p) \geq pf(q) + (1 - p) f(1 - q)$. Aequationes Math. **2**, 362–363 (1969).

3.9.74 D. S. MITRINOVIĆ [1] proposed the following problems:

Problem 1. Determine those algebraic functions $A_k(x)$, $k = 2, 3, \ldots$, which have the following properties:

$$\begin{aligned} &\frac{\log x}{x-1} \leq A_k(x) && (x > 0), \\ &A_k(x) \sim x^{-1/k} && (x \to 0+), \\ &xA_k(x) \sim x^{1/k} && (x \to +\infty), \\ &A_k(x) - \frac{\log x}{x-1} \sim a_k(x-1)^{2k-2} && (x \to 1), \end{aligned}$$

where a_k is independent of x.

J. KARAMATA [2] and D. BLANUŠA [3] have given the forms of $A_2(x)$, $A_3(x)$ and $A_4(x)$, namely:

$$A_2(x) = \frac{1}{\sqrt{x}}, \qquad A_3(x) = \frac{1 + \sqrt[3]{x}}{x + \sqrt[3]{x}},$$

$$A_4(x) = \frac{7 + 16t + 7t^2}{7t - t^2 + 18t^3 - t^4 + 7t^5}, \qquad \text{with } t = \sqrt[4]{x}.$$

In connection with functions A_2 and A_3, see 3.6.15 and 3.6.16.

Problem 2. Find also algebraic functions $A_k(x)$ $(k = 2, 3, \ldots)$ such that, for $x > 0$,

$$\frac{\log x}{x-1} \leq A_k(x) \quad \text{and} \quad A_m(x) \geq A_n(x) \quad \text{for} \quad 2 \leq m \leq n.$$

D. Blanuša [3] indicated, without proof, that

$$A_2(x) \geq A_3(x) \geq A_4(x),$$

for $x > 0$.

Remark. So far no solution has been published to the above problems.

References

1. Mitrinović, D. S.: Problem 5626. Amer. Math. Monthly **75**, 911–912 (1968).
2. Karamata, J.: Problem 1. Bull. Soc. Math. Phys. Serbie 1, No. 1, 77–78 (1949).
3. Blanuša, D.: Problem 13. Bull. Soc. Math. Phys. Serbie 1, No. 3–4, 156–157 (1949).

3.9.75 Let $x \mapsto T_n(x)$ be a real trigonometric polynomial all the roots of which are real, and let $\max\limits_{0 \leq x \leq 2\pi} |T_n(x)| = 1$.

In 1940, P. Erdös [1] conjectured that

$$\int_0^{2\pi} |T_n(x)|\, dx \leq 4.$$

Remark. It seems that this conjecture (E) has neither been proved nor disproved. W. K. Hayman on p. 27 of [2] has again in 1967 drawn attention to conjecture (E). H. Kuhn [3] has recently given two new conjectures from which follows the truth of (E).

References

1. Erdös, P.: Note on some elementary properties of polynomials. Bull. Amer. Math. Soc. **46**, 954–958 (1940).
2. Hayman, W. K.: Research Problems in Function Theory. London 1967.
3. Kuhn, H.: Über eine Vermutung von P. Erdös. Arch. Math. (Basel) (to appear).

3.9.76 Let $a, b > 0$. If $m, n > 1$, then

$$((a+b)^m - a^m)^n + ((a+b)^n - b^n)^m > (a+b)^{mn},$$

and if $0 < m, n < 1$ this inequality is reversed.

Let $a, b, c > 0$, $0 < p, q, r < 1$, $p + q + r = 1$, and $s = mnt$. If $m, n, t > 1$, then

$$((a+b+c)^{s/m} - a^{s/m})^m + ((a+b+c)^{s/n} - b^{s/n})^n + ((a+b+c)^{s/t} - c^{s/t})^t > 2\,(a+b+c)^s,$$

while if $0 < m, n, t < 1$, then this inequality is reversed.

Furthermore, for $\mu > 1$ and $\nu > 1$ we have

$$\prod_{i=1}^{\nu}\left(1 - \prod_{j=1}^{\mu} p_{ij}\right) + \prod_{j=1}^{\mu}\left(1 - \prod_{i=1}^{\nu} q_{ij}\right) > 1,$$

where

$$p_{ij} + q_{ij} = 1, \quad 0 < p_{ij} < 1 \quad (i = 1, \ldots, \nu;\ j = 1, \ldots, \mu).$$

Remark. The above results, due to J.C. TURNER, V. CONWAY, M.S. KLAMKIN and J. BRENNER, generalize the inequality of 3.6.36.

Reference

TURNER, J.C., and V. CONWAY: Problem 68-1. SIAM Review **11**, 402—406 (1969).

3.9.77 For the function f defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt \quad (x \geq 0)$$

J. T. CHU proved in [1] the following result: If f satisfies

$$\frac{1}{2}(1 - e^{-ax^2})^{1/2} \leq f(x) \leq \frac{1}{2}(1 - e^{-bx^2})^{1/2},$$

then it is necessary and sufficient that

$$0 \leq a \leq \frac{1}{2} \quad \text{and} \quad b \geq 2/\pi.$$

G. PÓLYA and J. D. WILLIAMS have earlier proved independently that

$$f(x) \leq \frac{1}{2}(1 - e^{-2x^2/\pi}).$$

Remark 1. The above results can be connected with MILLS' ratio (see 2.26).

Remark 2. References concerning results of G. PÓLYA and J. D. WILLIAMS are given in paper [1].

Reference

1. CHU, J.T.: On bounds for the normal integral. Biometrika **42**, 263—265 (1955).

3.9.78 Let $x \mapsto f(x)$ and $x \mapsto g(x)$ be nonnegative concave functions on $[0, a]$ $(a > 0)$ such that

$$0 < \left(\int_0^a f(x)^p\, dx\right)^{1/p} < +\infty \quad \text{and} \quad 0 < \left(\int_0^a g(x)^q\, dx\right)^{1/q} < +\infty.$$

Then:

1° For $p > 1$ and $q > 1$,

$$\frac{\int_0^a f(x)\, g(x)\, dx}{\left(\int_0^a f(x)^p\, dx\right)^{1/p} \left(\int_0^a g(x)^q\, dx\right)^{1/q}} \geq \frac{1}{6}(1 + p)^{\frac{1}{p}} (1 + q)^{\frac{1}{q}} a^{1 - \frac{1}{p} - \frac{1}{q}}.$$

2° For $|p| < 1$ and $|q| < 1$,

$$\frac{\int_0^a f(x)\,g(x)\,dx}{\left(\int_0^a f(x)^p\,dx\right)^{1/p}\left(\int_0^a g(x)^q\,dx\right)^{1/q}} \leq \frac{1}{3}(1+p)^{\frac{1}{p}}(1+q)^{\frac{1}{q}}\,a^{1-\frac{1}{p}-\frac{1}{q}}.$$

This result, due to D.C. BARNES, is in connection with Theorems 5 and 6 of 2.11.

Reference

BARNES, D.C.: Some complements of Hölder's inequality. J. Math. Anal. Appl. 26, 82–87 (1969).

3.9.79 The following inequality

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F > 0$$

holds for all real values of x and y if and only if either

$$1^\circ\ A > 0,\quad \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0,\quad \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} > 0,$$

or

$$2^\circ\ A > 0,\quad \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0,\quad \begin{vmatrix} A & B \\ D & E \end{vmatrix} = 0,\quad \begin{vmatrix} A & D \\ D & F \end{vmatrix} > 0,$$

or

$$3^\circ\ A = B = D = 0,\quad C > 0,\quad \begin{vmatrix} C & E \\ E & F \end{vmatrix} > 0,$$

or

$$4^\circ\ A = B = C = D = E = 0,\quad F > 0.$$

3.9.80 Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfy $0 \leq a_1 \leq \cdots \leq a_n$ and $0 \leq b_n \leq \cdots \leq b_1$ with

$$a_i \geq \frac{a_{i-1} + a_{i+1}}{2},\quad b_i \geq \frac{b_{i-1} + b_{i+1}}{2}\quad \text{for } i = 2, \ldots, n-1.$$

Then

$$\sum_{i=1}^n a_i b_i \geq \frac{n-2}{2n-1}\left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}.$$

The inequality is sharp and equality holds in the case $a_i = n - i$, $b_i = i - 1$ $(i = 1, \ldots, n)$.

This result was communicated to us by D. C. BARNES.

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Russian, Serbian and Bulgarian names are transcribed according to Mathematical Reviews.

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* This is the pseudonym of a group of French mathematicians which was "born" (founded) in 1933. Their fundamental treatise *Éléments de mathématique*, published in Paris, contains 35 volumes. Among the founders of the group Bourbaki there are H. Cartan (1904), J. Dieudonné (1906), and A. Weil (1906).

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